Unique Continuation Property of the Fractional Elliptic Operators and Applications

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Outline

- Part I: Landis conjecture (UCP at infinity)
  - (1) fractional Laplacian with a drift term; (2) general Schrödinger equation of fractional type; (3) Schrödinger equation with half Laplacian
  - **Main difficulties.** Proving UCP usually involves Carleman estimate, which works well for local operators. For non-local operator, we need some tricks.
- Part II: Localization of fractional elliptic operators
- Part III: General procedure of proving Landis conjecture and unique continuation property (UCP)
- Part IV: Landis conjecture for a special case - half Laplacian
- Part V: Strict monotonicity of eigenvalues and unique continuation property (UCP)

Part I: Landis Conjecture

What is Landis conjecture?

- We consider the following classical Schrödinger equation:

$$\Delta u + qu = 0 \quad \text{in } \mathbb{R}^n \tag{1}$$

  with $q \in L^\infty(\mathbb{R}^n)$.
- Let $u \in \mathcal{C}^2(\mathbb{R}^n)$ such that $u(x) = \exp(-|x|)$ for all $|x| > 1$.
- There exists a constant $C > 0$ such that $|\Delta u| \leq C^2 |u|$ in $\mathbb{R}^n$, that is, u satisfies (1) with

$$q(x) = \begin{cases} -\dfrac{\Delta u(x)}{u(x)} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

- By scaling, we can make $|q| \leq 1$: Precisely, if we let $u_C(x) = u(Cx)$, then $|\Delta u_C| \leq |u_C|$.

What is Landis conjecture?

- **An example.** We can choose a constant $C > 0$ and construct a function $u \in \mathcal{C}^2(\mathbb{R}^n)$ such that

$$\begin{cases} u(x) = \exp(-C|x|) & \text{for all } |x| > 1, \\ |\Delta u| \leq |u| & \text{in } \mathbb{R}^n. \end{cases}$$

Conjecture (Landis 60's)

Let $|q(x)| \leq 1$*. If* $|u(x)| \leq C_0$ *satisfies* $\Delta u + qu = 0$ *in* $\mathbb{R}^n$*, and*

$$|u(x)| \leq \exp(-C|x|^\alpha) \quad \textit{for some } \alpha > 1,$$

then $u \equiv 0$.

- Unique continuation property at infinity.

Some history of Landis conjecture

- **Landis conjecture.** $u \in L^\infty(\mathbb{R}^n)$ satisfies Schrödinger equation. If $|u(x)| \leq \exp(-C|x|^{1+})$, then $u \equiv 0$.
- This conjecture was disproved by Meshkov in 1991:
  - Constructed complex-valued potential q and $u \not\equiv 0$ with
  $$|u(x)| \leq \exp(-C|x|^{\frac{4}{3}}).$$
  - If $|u(x)| \leq \exp(-C|x|^\alpha)$ for some $\alpha > \frac{4}{3}$, then $u \equiv 0$.
- (Bourgain-Kenig '05) Proved Meshkov's result in quantitative form.
  - Carleman estimate (cannot distinguish real or complex q and u)
- See (Davey '14) and (Lin-Wang '14) for Quantative Landis conjecture with drift term.
- (Kenig '06) Refined the conjecture for real-valued potentials q and u.
  - Partial answers: (Davey-Kenig-Wang '17 '19), (Rossi '18), (Loguv-Mallinnikova-Nadirashvili-Nazarov '20)

Fractional Landis conjecture

- Let $s \in (0,1)$, and we consider the fractional Schrödinger equation

$$(-\Delta)^s u + qu = 0 \quad \text{in } \mathbb{R}^n,$$

where $\mathscr{F}((-\Delta)^s u) := |\xi|^{2s}\hat{u}(\xi)$ and $\mathscr{F}$ is Fourier transform.

- Both qualitative and quantative Landis conjecture proved in (Rüland-Wang '19).
  - Let q is differentiable with $|x \cdot \nabla q(x)| \leq 1$:

$$\text{If } \int_{\mathbb{R}^n} e^{|x|^\beta} |u|^2 \, dx < \infty \text{ for some } \beta > 1, \text{ then } u \equiv 0.$$

  - For non-differentiable q, we consider $s \in (\frac{1}{4}, 1)$:

$$\text{If } \int_{\mathbb{R}^n} e^{|x|^\beta} |u|^2 \, dx < \infty \text{ for some } \beta > \frac{4s}{4s-1}, \text{ then } u \equiv 0.$$

- $\frac{4s}{4s-1} \to \frac{4}{3}$ as $s \to 1$.

Fractional Laplacian with a drift term

- Fractional Schrödinger equation with a drift term:

$$((-\Delta)^s + b(x)x \cdot \nabla + q(x))u = 0 \quad \text{in } \mathbb{R}^n, \tag{2}$$

where b and q are scalar-valued functions. Motivated by (Rossi '18), we consider the drift term only involving the radial derivative.

Theorem (Ghosh-Salo-Uhlmann '20)

If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$ and $u = (-\Delta)^s u = 0$ in some open set $W \subset \mathbb{R}^n$, then $u \equiv 0$.

Corollary (Unique continuation property)

Let u be a solution to (2). If $u = 0$ in some open set $W \subset \mathbb{R}^n$, then $u \equiv 0$.

- This dissertation is the first attempt to study the Landis conjecture (UCP at infinity) of (2).

Fractional Laplacian with a drift term

Theorem (Dissertation: Differentiable potential)

Let $n = 1, 2, 3$. Let $s \in (\frac{1}{2}, 1)$ when $n = 1, 2$, and let $s \in (\frac{3}{4}, 1)$ when $n = 3$. We assume that there exists a constant λ such that

$$|q(x)| \leq \lambda \text{ and } 0 \leq b(x) \leq \lambda |x|^{-\beta} \text{ for all } x \in \mathbb{R}^n,$$

and the radial derivatives of q, b satisfy

$$|x \cdot \nabla q(x)| \leq \lambda \text{ and } |x \cdot \nabla b(x)| \leq \lambda |x|^{-\beta} \text{ for all } x \in \mathbb{R}^n$$

for some $\beta > 1$. If $u \in H^s(\mathbb{R}^n)$ is a solution such that

$$\int_{\mathbb{R}^n} e^{|x|^\beta} \left[|u(x)|^2 + |\nabla u(x)|^2 \right] dx \leq \lambda,$$

then $u \equiv 0$.

Fractional Laplacian with a drift term

Theorem (Dissertation: Non-differentiable potential)

Let $n = 1, 2, 3$. Let s be given in previous theorem. We assume that there exists a constant λ such that

$$|q(x)| \leq \lambda \text{ and } 0 \leq b(x) \leq \lambda |x|^{-\beta} \text{ for all } x \in \mathbb{R}^n,$$

and the radial derivatives of b satisfy

$$|x \cdot \nabla b(x)| \leq \lambda |x|^{-\beta} \text{ for all } x \in \mathbb{R}^n$$

for some $\beta > \frac{4s}{4s-1}$. If $u \in H^s(\mathbb{R}^n)$ is a solution such that

$$\int_{\mathbb{R}^n} e^{|x|^\beta} \left[|u(x)|^2 + |\nabla u(x)|^2 \right] dx \leq \lambda,$$

then $u \equiv 0$.

Extension to fractional elliptic operators

Lemma (Bochner's formula)

Let $s \in (0,1)$, *and define* $\Gamma(-s) := \frac{1}{-s}\Gamma(1-s)$, *then*

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1)u(x)\,\frac{dt}{t^{1+s}} \quad \textit{for all } \lambda > 0, \tag{3}$$

- See e.g. (Schilling-Song-Vondracek '10).
- Let P be a second order elliptic operator in divergence form:

$$P = \nabla \cdot A\nabla = \sum_{j,k=1}^{n} \partial_j a_{jk}(x)\partial_k.$$

Generalized Schrödinger equation of fractional type

- Since $-P$ is a non-negative operator, by formally replacing λ by $-P$ in Bochner's formula, this also suggests us to define

$$(-P)^s u(x) := \frac{1}{\Gamma(-s)} \int_0^\infty (e^{tP} - 1) u(x) \, \frac{dt}{t^{1+s}},$$

where $\{e^{tP}\}_{t \geq 0}$ is the semi-group generated by $-P$, see (Stinga-Torrea '10).

- Generalized Fractional Schrödinger equation:

$$((-P)^s + q)u = 0 \quad \text{in } \mathbb{R}^n$$

with $s \in (0,1)$ and $|q(x)| \leq 1$.

Generalized Schrödinger equation of fractional type

- **Ellipticity condition.** There exists a constant $0 < \lambda < 1$ such that

$$\lambda|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \leq \lambda^{-1}|\xi|^2 \quad \text{for all } x \in \mathbb{R}^n.$$

- **Regularity and symmetry.** $a_{jk} = a_{kj}$ are Lipschitz.
- $A \approx \mathrm{Id}$ **(i.e.** $P \approx \Delta$**) at infinity.** There exists a constant $C > 0$ and a sufficiently small parameter $\epsilon > 0$ such that

$$\max_{1\leq j,k\leq n} \sup_{|x|\geq 1} |a_{jk}(x) - \delta_{jk}(x)| + \max_{1\leq j,k\leq n} \sup_{|x|\geq 1} |x||\nabla a_{jk}(x)| \leq \epsilon, \tag{4a}$$

$$\max_{1\leq j,k\leq n} \sup_{|x|\geq 1} |\nabla^2 a_{jk}(x)| \leq C. \tag{4b}$$

- When $s = \frac{1}{2}$, we no need to assume (4b).

Generalized Schrödinger equation of fractional type

Theorem (Dissertation: Differentiable potential)

Let $s \in (0,1)$ *and assume that* $u \in H^s(\mathbb{R}^n)$ *is a solution. We assume that the potential* $q \in C^1(\mathbb{R}^n)$ *satisfies* $|q(x)| \leq 1$ *and*

$$|x||\nabla q(x)| \leq 1.$$

If u *satisfies*

$$\int_{\mathbb{R}^n} e^{|x|^\beta} |u|^2 \, dx \leq C < \infty \quad \text{for some } \beta > 1,$$

then $u \equiv 0$.

Generalized Schrödinger equation of fractional type

Theorem (Dissertation: Non-differentiable potential)

Let $s \in (\frac{1}{4}, 1)$ and assume that $u \in H^s(\mathbb{R}^n)$ is a solution. We assume that the potential $|q(x)| \leq 1$. If u satisfies

$$\int_{\mathbb{R}^n} e^{|x|^\beta} |u|^2 \, dx \leq C < \infty$$

for some $\beta > \frac{4s}{4s-1}$, then $u \equiv 0$.

- Using Fourier transform, it is easy to see that $(-\Delta)^\alpha (-\Delta)^\beta = (-\Delta)^{\alpha+\beta}$, and $(-\Delta)^s : \dot{H}^{\beta+s}(\mathbb{R}^n) \to \dot{H}^{\beta-s}(\mathbb{R}^n)$ is bounded for all $\beta \in \mathbb{R}$.
- However, extension of these properties to $(-P)^s$ is not trivial.
- For the case when $a_{jk} \in \mathcal{C}^\infty$, $(-P)^s$ is a pseudo-differential operator of order $2s$, see (Seeley '67).

A special case: Schrödinger equation with half-Laplacian

- Let $|q| \leq 1$ and let $u \in H^{\frac{1}{2}}(\mathbb{R}^n)$ be a solution to $(-\Delta)^{\frac{1}{2}} u + qu = 0$.
- (Rüland-Wang '19) If there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} e^{|x|^\beta} |u|^2 \, dx \leq C < \infty \quad \text{for some } \beta > 2,$$

then $u \equiv 0$.

Theorem (Dissertation: Improvement)

If there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} e^{|x|} |u|^2 \, dx \leq C < \infty,$$

then $u \equiv 0$.

- Here, the potential q need not to be real-valued. It is interesting to compare this result with the real-version of the Landis-type conjecture.

Part II: Localization of fractional elliptic operators

Half Laplacian

- We now introduce an equivalent definition of $(-\Delta)^s$ on $\mathbb{R}^n$.
- **(Kwaśnicki '17).** There are at least 10 equivalent definitions.
- To motivate the ideas, here we perform some formal computations.
- In order to make things easy, we first consider $s = 1/2$.
- Write $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \mathbb{R}_{>0}$. For a function $u : \mathbb{R}^n \to \mathbb{R}$, we consider a function $\tilde{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ satisfies

$$\begin{cases} \Delta \tilde{u} = \Delta_{x'} \tilde{u} + \partial_{n+1}^2 \tilde{u} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Half Laplacian

- Taking Fourier transform with respect to variable x', we reach

$$\begin{cases} -|\xi|^2 \hat{\tilde{u}} + \partial_{n+1}^2 \hat{\tilde{u}} = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \hat{\tilde{u}} = \hat{u} & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

  where $\hat{\tilde{u}} = \mathscr{F}\tilde{u}$ and $\hat{u} = \mathscr{F}u$.

- Plugging the special solution $\hat{\tilde{u}}(\xi, x_{n+1}) = \hat{u}(\xi)\phi(z)$ with $z = |\xi|x_{n+1}$, we obtain

$$\begin{cases} -\phi(z) + \partial_z^2\phi(z) = 0 & \text{for } z > 0, \\ \phi(0) = 1 & (\text{we additional assume } \lim_{z\to\infty}\phi(z) = 0 \text{ and } \phi \in \mathcal{C}^0). \end{cases}$$

- The unique solution (with additional conditions) is given by $\phi(z) = e^{-z}$.
- We obtain a special solution $\hat{\tilde{u}}(\xi, x_{n+1}) = \hat{u}(\xi)e^{-|\xi|x_{n+1}}$.

Half Laplacian

- Since $\hat{\tilde{u}}(\xi, x_{n+1}) = \hat{u}(\xi)e^{-|\xi|x_{n+1}}$, we have

$$\lim_{x_{n+1}\to 0} \partial_{n+1}\hat{\tilde{u}}(\xi, x_{n+1}) = -|\xi|\hat{u}(\xi).$$

- In view of Fourier definition, it is make sense to define

$$\begin{cases} \Delta\tilde{u} = \Delta_{x'}\tilde{u} + \partial_{n+1}^2\tilde{u} = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \tilde{u} = u & \text{on } \mathbb{R}^n \times \{0\}. \\ -(-\Delta)_{\mathrm{CS}}^{1/2}u(x') := \lim_{x_{n+1}\to 0}\partial_{n+1}\tilde{u}(x) & \text{for all } x' \in \mathbb{R}^n. \end{cases}$$

- The half-Laplacian can be defined in terms of Dirichlet-to-Neumann map (DN-map) of a harmonic functions in half space $\mathbb{R}_+^{n+1}$.

Fractional Laplacian of order $0 < s < 1$

- We now perform the similar formal computations for $(-\Delta)^s$ with $0 < s < 1$.
- For a function $u : \mathbb{R}^n \to \mathbb{R}$, we consider a function $\tilde{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ satisfies

$$\begin{cases} \nabla \cdot x_{n+1}^{1-2s} \nabla \tilde{u} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{5}$$

- Note that (5) is equivalent to

$$\begin{cases} \Delta_{x'} \tilde{u} + \dfrac{1-2s}{x_{n+1}} \partial_{n+1} \tilde{u} + \partial_{n+1}^2 \tilde{u} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Fractional Laplacian of order $0 < s < 1$

- Taking Fourier transform with respect to variable x', we reach

$$\begin{cases} -|\xi|^2\hat{\tilde{u}} + \dfrac{1-2s}{x_{n+1}}\partial_{n+1}\hat{\tilde{u}} + \partial_{n+1}^2\hat{\tilde{u}} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \hat{\tilde{u}} = \hat{u} & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

  where $\hat{\tilde{u}} = \mathscr{F}\tilde{u}$ and $\hat{u} = \mathscr{F}u$.
- Plugging the special solution $\hat{\tilde{u}}(\xi, x_{n+1}) = \hat{u}(\xi)\phi(z)$ with $z = |\xi|x_{n+1}$, we obtain

$$\begin{cases} -\phi(z) + \dfrac{1-2s}{z}\partial_z\phi(z) + \partial_z^2\phi(z) = 0 \quad \text{for } z > 0, \\ \phi(0) = 1 \quad (\text{we additional assume } \lim_{z\to\infty}\phi(z) = 0 \text{ and } \phi \in \mathcal{C}^0) \end{cases}$$

- The unique solution (with additional conditions) is given by

$$\phi(z) = \frac{2^{1-s}}{\Gamma(s)} z^s K_s(z) \quad , K_s = \text{modified Bessel function of } 2^{\text{nd}} \text{ kind.}$$

Fractional Laplacian of order $0 < s < 1$

- Using the properties of the modified Bessel functions, we have

$$-\lim_{z\to 0_+} z^{1-2s}\partial_z\phi(z) = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}.$$

- Hence, by writing $c_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)} > 0$ (indeed, $c_{1/2} = 1$), we have

$$c_s \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\hat{\tilde{u}}(\xi, x_{n+1}) = -|\xi|^{2s}\hat{u}(\xi).$$

- In view of Fourier definition, it is make sense to define

$$\begin{cases} \nabla\cdot x_{n+1}^{1-2s}\nabla\tilde{u} = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \tilde{u} = u & \text{on } \mathbb{R}^n\times\{0\}. \\ -(-\Delta)_{\mathrm{CS}}^s u(x') := c_s \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}(x) & \text{for all } x'\in\mathbb{R}^n. \end{cases}$$

- $(-\Delta)^s$ can be defined in terms of DN-map of a degenerate elliptic equation in half space $\mathbb{R}_+^{n+1}$.

Fractional Laplacian of order $0 < s < 1$

Theorem (Caffarelli-Silvestre '07)

Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$. Let $(-\Delta)^s_{\mathrm{F}}$ be the fractional Laplacian defined via Fourier transform. Let

$$\tilde{u} \in \dot{H}^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s}) := \left\{ v : \mathbb{R}^{n+1}_+ \to \mathbb{R} \;\middle|\; \int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2s} |\nabla v|^2 \, dx < \infty \right\}$$

be a solution to the extension problem (which is a degenerate elliptic equation). Then $(-\Delta)^s_{\mathrm{F}} = (-\Delta)^s_{\mathrm{CS}}$.

- $(-\Delta)^s_{\mathrm{F}}$ is defined via Fourier transform, which is non-local.
- $(-\Delta)^s_{\mathrm{CS}}$ is defined via a local (but degenerate) elliptic equation.
- Using this equivalent definition, we can obtain Carleman estimate in the extended space $\mathbb{R}^{n+1}_+$ rather than the original $\mathbb{R}^n$.

General elliptic operator of fractional type

- The localization technique of $(-\Delta)^s$ also works for Bochner elliptic operator $(-P)^s$.
- For $s \in (0,1)$, we consider a solution $\tilde{u}$ of the degenerate elliptic equation

$$\begin{aligned} \left[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P\right] \tilde{u} &= 0 \quad \text{in } \mathbb{R}_+^{n+1}, \\ \tilde{u} &= u \quad \text{on } \mathbb{R}^n \times \{0\}. \end{aligned}$$

- (Stinga-Torrea '10) The fractional operator $(-P)^s$ (defined via Bochner's formula) satisfies

$$-(-P)^s u(x') = c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(x)$$

for some $c_{n,s} > 0$. In fact, if $s = \frac{1}{2}$, then $c_{n,s} = 1$.

Part III: General procedure of proving Landis conjecture and unique continuation property (UCP)

- We now sketch the general procedure of proving **Landis conjecture** and **unique continuation property (UCP)** for fractional elliptic equation.
- To explain this general procedure, as an example, we here consider Landis conjecture for general Schrödinger equation of fractional type.
- We consider the fractional Schrödinger equation

$$((-P)^s + q)u = 0 \quad \text{in } \mathbb{R}^n,$$

where $P = \nabla \cdot A\nabla$ satisfies some conditions, and $P \approx \Delta$ at infinity.

Step 1: Localization

- Using the Caffarelli-Silvestre type extension, we can localize $((-P)^s + q)u = 0$ as the following:

$$\begin{cases} \left[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P\right] \tilde{u} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on } \mathbb{R}^n \times \{0\}, \\ c_{n,s} \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(x) = q(x')u & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

  where we recall $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+ = \mathbb{R}^{n+1}_+$.
- We called $\tilde{u}$ the Caffarelli-Silvestre type extension of u.

Step 2: Boundary decay implies bulk decay

Proposition

Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$ be a solution to $((-P)^s + q)u = 0$, with $|q(x)| \leq 1$ and some appropriate assumptions. Assume that $P \approx \Delta$ at infinity. If there exists $\alpha > 1$ such that

$$\int_{\mathbb{R}^n} e^{|x|^\alpha} |u|^2 \, dx \leq C < \infty, \tag{6}$$

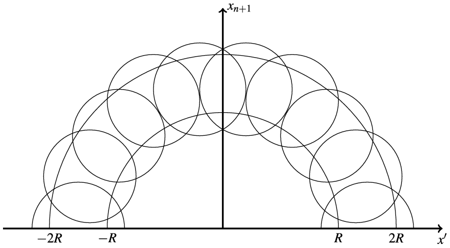
then there exist constants $C_1, C_2 > 0$ so that the Caffarelli-Silvestre type extension $\tilde{u}$ of u satisfies

$$|\tilde{u}(x)| \leq C_1 e^{-C_2 |x|^\alpha} \quad \text{for all } x \in \mathbb{R}^{n+1}_+. \tag{7}$$

- Here, we remark that both (6) and (7) decay at the same rate $\alpha > 1$.
- This enables us to obtain a Carleman estimate in the extension $\mathbb{R}^{n+1}_+$.

Step 2: Boundary decay implies bulk decay

- **Idea.** Propagation of smallness. The extension problem is simply an elliptic equation in $\{x_{n+1} > c\}$, therefore, we have 3-ball inequality.
- We only need to pass the boundary decay on $\mathbb{R}^n \times \{0\}$ to a small neighborhood. This technical part relies on a delicate Carleman estimate.



Source: (Rüland-Wang '19)

Step 3: Carleman estimate

Theorem (Carleman estimate for non-differentiable potential)

Let $a_{jk} \approx \delta_{jk}$ *at infinity (as well as other assumptions). Let* $s \in (0,1)$ *and let* $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ *with* $\operatorname{supp}(\tilde{u}) \subset \mathbb{R}^{n+1}_+ \setminus B_1^+$ *be a solution to*

$$\left[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-s} \sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k\right] \tilde{u} = f \quad \text{in } \mathbb{R}^{n+1}_+,$$

$$\lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} = V \tilde{u} \quad \text{on } \mathbb{R}^n \times \{0\}.$$

Let $\phi(x) = |x|^\alpha$ *for* $\alpha \geq 1$*. Then* $\exists C > 0$ *such that for all* $\tau \gg 1$ *we have*

$$\tau^3 \| e^{\tau\phi} |x|^{\frac{3\alpha}{2}-1} x_{n+1}^{\frac{1-2s}{2}} \tilde{u} \|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \| e^{\tau\phi} |x|^{\frac{\alpha}{2}} x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u} \|^2_{L^2(\mathbb{R}^{n+1}_+)}$$
$$\leq C \left[\| e^{\tau\phi} x_{n+1}^{\frac{2s-1}{2}} |x| f \|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau^{2-2s} \| e^{\tau\phi} V |x|^{(1-\alpha)s} \tilde{u} \|^2_{L^2(\mathbb{R}^n \times \{0\})} \right].$$

Step 4: Trace estimate

- Unlike the classical Carleman estimate on $\mathbb{R}^n$, there is a boundary term in the Carleman estimate on half-space $\mathbb{R}^{n+1}_+$:

$$\tau^3\|e^{\tau\phi}|x|^{\frac{3\alpha}{2}-1}x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)}^2+\tau\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)}^2$$
$$\leq C\left[\|e^{\tau\phi}x_{n+1}^{\frac{2s-1}{2}}|x|f\|_{L^2(\mathbb{R}^{n+1}_+)}^2+\tau^{2-2s}\|e^{\tau\phi}V|x|^{(1-\alpha)s}\tilde{u}\|_{L^2(\mathbb{R}^n\times\{0\})}^2\right].$$

- We need the following trace estimate to "absorb" the boundary term.

Lemma (Rüland '15)

Let $\sigma\in(0,1)$. Then there exists constant $C=C(n,\sigma)>0$ such that

$$\|v\|_{L^2(\partial\mathcal{S}^n_+)}\leq C\left[\beta^{1-\sigma}\|\theta_{n+1}^{\frac{1-2s}{2}}v\|_{L^2(\mathcal{S}^n_+)}+\beta^{-\sigma}\|\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}v\|_{L^2(\mathcal{S}^n_+)}\right]$$

for all $\beta>1$, where $\theta=(\theta_1,\cdots,\theta_n,\theta_{n+1})\in\mathcal{S}^n_+$, $\nabla_{\mathcal{S}^n}=(\Theta_1,\cdots,\Theta_n,\Theta_{n+1})$, and Θ_k are vector fields on $\mathcal{S}^n$.

Step 4: Trace estimate

- Write the trace estimate as

$$\beta^{2-2\sigma}\|v\|_{L^2(\partial\mathcal{S}^n_+)}^2 \leq C\left[\beta^{4-4\sigma}\|\theta_{n+1}^{\frac{1-2s}{2}} v\|_{L^2(\mathcal{S}^n_+)}^2 + \beta^{2-4\sigma}\|\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n} v\|_{L^2(\mathcal{S}^n_+)}^2\right].$$

- Indeed, we shall choose $\beta \approx \tau$ (up to some suitable multiplicative constant), where τ is the parameter in the Carleman estimate

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\alpha}{2}-1}x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \tau\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \\
&\leq C\left[\|e^{\tau\phi}x_{n+1}^{\frac{2s-1}{2}}|x|f\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \tau^{2-2s}\|e^{\tau\phi}V|x|^{(1-\alpha)s}\tilde{u}\|_{L^2(\mathbb{R}^n\times\{0\})}^2\right].
\end{aligned}$$

- The boundary term for large $\tau \gg 1$, when $s > \frac{1}{4}$.
- Finally, we can prove $\tilde{u} \equiv 0$ using a contradiction argument, and hence $u = \tilde{u}|_{\mathbb{R}^n\times\{0\}} \equiv 0$.

Part IV: Landis conjecture for a special case - half Laplacian

Special structures of half-Laplacian

- Using the Caffarelli-Silvestre extension, we can reformulate the equation $(-\Delta)^{1/2}u + qu = 0$ in $\mathbb{R}^n$ as

$$\begin{cases} \Delta\tilde{u} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on } \mathbb{R}^n \times \{0\}, \\ \lim_{x_{n+1}\to 0} \partial_{n+1}\tilde{u} = qu & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

- Since $\tilde{u}$ is harmonic in $\mathbb{R}^{n+1}_+$, this suggests us to introduce a conformal mapping from the ball to the upper half-space, and back.
- **Idea.** Finding a mapping which preserves Laplacian.

Conformal mapping

- For each $x \in \mathbb{R}^{n+1} \setminus \{0\}$, we define $x^* := x/|x|^2$, i.e. the inverse relative to the unit sphere $\mathcal{S}^n$.
- Let $s = (0, \cdots, 0, -1)$ be the south pole of $\mathcal{S}^n$, and we define $\Phi : \mathbb{R}^{n+1} \setminus \{s\} \to \mathbb{R}^{n+1} \setminus \{s\}$ by

$$\Phi(z) := 2(z - s)^* + s.$$

- We also can regard Φ as a homeomorphism from $\mathbb{R}^n \cup \{\infty\}$ (one-point compactification) onto itself by defining $\Phi(s) = \infty$ and $\Phi(\infty) = s$.

Lemma (see e.g. Axler-Bourdon-Ramey '92)

$\Phi : \mathbb{R}^{n+1} \setminus \{s\} \to \mathbb{R}^{n+1} \setminus \{s\}$ is injective. Furthermore, it maps $B_1(0)$ onto $\mathbb{R}^{n+1}_+$, and also maps $\mathbb{R}^{n+1}_+$ onto $B_1(0)$.

Conformal mapping

- Given any function w defined on a domain $\Omega \subset \mathbb{R}^{n+1} \setminus \{s\}$, the Kelvin transform of w, which is the function $\mathcal{K}[w]$ on $\Phi(\Omega)$, is defined by

$$\mathcal{K}[w](z) := 2^{\frac{n-1}{2}} |z - s|^{1-n} w(\Phi(z)).$$

Lemma (see e.g. Axler-Bourdon-Ramey '92)

Let Ω be any domain in $\mathbb{R}^{n+1} \setminus \{s\}$. Then w is harmonic on Ω if and only if $\mathcal{K}[w]$ is harmonic on $\Phi(\Omega)$.

Main ideas

Sketch of the proof. Since $\Delta \tilde{u} = 0$ in $\mathbb{R}^{n+1}_{+}$, then $\Delta(\mathcal{K}[\tilde{u}]) = 0$ in $B_1(0)$. Using the boundary to bulk decay result above, the boundary decay $\int_{\mathbb{R}^n} e^{|x|}|u|^2\,dx \leq C$ implies bulk decay $|\tilde{u}(x)| \leq Ce^{-|x|}$. Indeed,

$$|\mathcal{K}[\tilde{u}](z)| \leq C \exp\left(-\frac{c}{|z-s|}\right) \quad \text{near the south pole } s.$$

Since we can extend $\mathcal{K}[\tilde{u}]$ on $\overline{B_1(0)}$, using a result in (Jin '93), we conclude that $\mathcal{K}[\tilde{u}] \equiv 0$, therefore, we conclude $\tilde{u} \equiv 0$ (hence $u = \tilde{u}|_{\mathbb{R}^n \times \{0\}} \equiv 0$). $\square$

- If we employ the ideas for general $(-\Delta)^s$ or $(-P)^{1/2}$, indeed $\mathcal{K}[\tilde{u}]$ satisfies an elliptic equation on $B_1(0)$. However, in this case, it cannot be extended to $\overline{B_1(0)}$ (precisely, at the south pole s).
- After we transform the decay from boundary to bulk, the proof uses conformal geometry rather than Carleman estimate, which even does not depend $\tilde{u}|_{\mathbb{R}^n \times \{0\}}$. Therefore, it doesn't matter whether q is real-valued or not.

Part V: Strict monotonicity of eigenvalues and unique continuation property (UCP)

Some results for elliptic operators

- Let Ω be a Lipschitz domain in $\mathbb{R}^n$, and let A be a second order elliptic operator given by

$$Au := -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + a_0(x)u$$

  with $a_0(x) \geq 0$, $a_0 \in L^r(\Omega)$ for some $r > \frac{n}{2}$, and $(a_{ij}(x)) \in L^\infty(\Omega)$ is symmetric and satisfies the elliptic condition.
- Given a weight function $m \in L^r(\Omega)$, where the exponent r is given above, we consider the eigenvalue problem

$$\begin{cases} Au = \mu m(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

- It is known that the eigenvalues of (8), depending on m, form a countable sequence:

$$\cdots \leq \mu_{-2}(m) \leq \mu_{-1}(m) < 0 < \mu_1(m) \leq \mu_2(m) \leq \cdots .$$

Some results for elliptic operators

- If m is non-negative (resp. non-positive), then the sequence of eigenvalues is bounded below (resp. bounded above).
- In fact, by using a variational characterization of eigenvalues, we can observe that each μ_k is non-increasing in the weight function m.
  - That is, if $m(x) \leq \hat{m}(x)$ a.e., then $\mu_k(\hat{m}) \leq \mu_k(m)$.
- (Figueiredo-Goessez '92) $\mu_k(m)$ is strictly decreasing in $m \iff$ the corresponding eigenfunction enjoys the unique continuation property from a set of positive measure (a.k.a. measurable unique continuation property, MUCP).
- We say that $u_k(x)$ has the **MUCP**, if $u = 0$ in $E \subset \Omega$ with $|E| > 0$, then $u \equiv 0$ in Ω.
- (Tsouli-Chakrone-Rahmani-Darhouche '12) A similar result was proved for the bi-harmonic operator $(-\Delta)^2$.
- (Frassu-Iannizzoto '20) The equivalence of strict monotonicity and MUCP was further extended to some non-local operators.

Main result

- In this dissertation, we established the equivalence of strict monotonicity of eigenvalues and measurable unique continuation property (MUCP) for the spectral elliptic operator $(-P)^{\gamma}$, where $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$.

Definition of spectral fractional Laplacian

- We first explain the definition of $(-P)^{\gamma}$ for $\gamma = s \in (0,1)$.
- **Recall.** It is known that the eigenvalues of $-Pu = \lambda u$ in Ω for $u \in H_0^1(\Omega)$ form a countable sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$.
- Let $\{\phi_k\}_{k=1}^{\infty}$ be the corresponding eigenfunctions, and we have

$$-Pu = \sum_{k=1}^{\infty} \lambda_k u_k \phi_k \quad \text{provided} \quad u = \sum_{k=1}^{\infty} u_k \phi_k \in H_0^1(\Omega).$$

- We simply define $(-P)^s u := \sum_{k=1}^{\infty} \lambda_k^s u_k \phi_k$ for $u \in \mathrm{dom}\,((-P)^s)$.
  - Here, the spectral elliptic operator $(-P)^s$ is not the restriction of the Bochner elliptic operator $(-P)^s$ for $\mathbb{R}^n$ on Ω.
  - Moreover, the spectral elliptic operator $(-P)^s$ does not included in (Frassu-Iannizzoto '20).

Eigenvalue problem for Bi-harmonic operator

- It is natural to define $(-P)^\gamma$ for $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$ using a similar manner as $(-P)^s$. However, we need to impose some suitable boundary conditions.
- Bi-harmonic operator with Dirichlet boundary condition:

$$\begin{cases} (-\Delta)^2 u = \lambda_{\mathrm{D}} u & \text{in } \Omega, \\ u = 0, \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

- Bi-harmonic operator with Navier boundary condition:

$$\begin{cases} (-\Delta)^2 u = \lambda_{\mathrm{N}} u & \text{in } \Omega, \\ u = 0, -\Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

- Existence of eigenvalues and eigenfunctions of both problems above are known.
- We will impose the Navier boundary condition for $(-P)^\gamma$.

Definition of spectral fractional Laplacian

- Assuming $(a_{ij}) \in \mathcal{C}^{2\lfloor\gamma\rfloor,1}(\Omega)$, where $\lfloor\gamma\rfloor$ is the integer part of γ.
- Let $H^\alpha(\Omega)$ be the restriction of $H^\alpha(\mathbb{R}^n)$ to Ω, and let $\tilde{H}^\alpha(\Omega)$ be:
  - When $0 < \alpha < 1/2$, $\tilde{H}^\alpha(\Omega) := H^\alpha(\Omega)$.
  - When $1/2 < \alpha < 5/2$, $\tilde{H}^\alpha(\Omega) := \left\{ H^\alpha(\Omega) \mid u = 0 \text{ on } \partial\Omega \right\}$.
  - When $2k + 1/2 < \alpha < 2k + 5/2$,

$$\tilde{H}^\alpha(\Omega) := \left\{ H^\alpha(\Omega) \mid u = \cdots = (-P)^k u = 0 \text{ on } \partial\Omega \right\}.$$

  - When $\alpha = 2k + 1/2$,

$$\tilde{H}^\alpha(\Omega) := \left\{ H^\alpha(\Omega) \;\middle|\; \begin{array}{l} u = \cdots = (-P)^{k-1} u = 0 \text{ on } \partial\Omega \\ (-P)^\alpha u \in H^{1/2}(\mathbb{R}^n) \\ \mathrm{supp}\,((-P)^\alpha u) \subset \overline{\Omega} \end{array} \right\}.$$

- $\mathrm{dom}\,((-P)^\gamma) := \tilde{H}^{2\gamma}(\Omega)$, and $(-P)^\gamma : \tilde{H}^{2\gamma}(\Omega) \to L^2(\Omega)$.

A properties of spectral fractional Laplacian

- We now exhibit a very basic but important properties of $(-P)^\gamma$.

Lemma

Let $\lfloor\gamma\rfloor$ be the integer part of γ and $s := \gamma - \lfloor\gamma\rfloor$. If $u \in \mathrm{dom}\,((-P)^\gamma) = \tilde{H}^{2\gamma}(\Omega)$, then

$$(-P)^\gamma u = (-P)^s((-P)^{\lfloor\gamma\rfloor} u) = (-P)^{\lfloor\gamma\rfloor}((-P)^s u).$$

- The proof is easy. However, here we want to point out that the Navier boundary condition in $\tilde{H}^{2\gamma}(\Omega)$ is essential in the proof.
- In fact,

$$(-P)^s u := \sum_{k=1}^{\infty} \lambda_k^s u_k \phi_k \quad \text{provided } u = \sum_{k=1}^{\infty} u_k \phi_k$$

can be simply define without Navier boundary condition. However, the lemma cannot hold in this case.
- $(-P)^\gamma$ also called the Navier fractional elliptic operator.

Some remarks on UCP

- The localization technique of $(-\Delta)^s$ (as well as Bochner elliptic operator $(-P)^s$ in $\mathbb{R}^n$) also works for spectral elliptic operator $(-P)^s$ in Ω.
- For $s \in (0,1)$, we consider a solution $\tilde{u}$ of the degenerate elliptic equation

$$\begin{aligned} \left[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P\right] \tilde{u} &= 0 \quad \text{in } \Omega \times (0,\infty), \\ \tilde{u} &= u \quad \text{on } \Omega \times \{0\}, \\ \tilde{u} &= 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned}$$

- (GarcíaFerrero-Rüland '19) The fractional operator $(-P)^s$ (defined via Bochner's formula) satisfies

$$(-P)^s u(x') = c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(x)$$

  for some $c_{n,s} \neq 0$.
- There is also a corresponding extension problem for $(-P)^\gamma$ for each $\gamma \in \mathbb{R} \setminus \mathbb{N}$.

Some remarks on UCP

- Formally, the extension problem for spectral elliptic operator is very similar to the one for Bochner elliptic operator:
  - Extension problem for Bochner elliptic operator

$$\begin{aligned} \left[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P\right] \tilde{u} &= 0 \quad \text{in } \mathbb{R}_{+}^{n+1}, \\ \tilde{u} &= u \quad \text{on } \mathbb{R}^{n} \times \{0\}. \end{aligned}$$

  - Extension problem for spectral elliptic operator

$$\begin{aligned} \left[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P\right] \tilde{u} &= 0 \quad \text{in } \Omega \times (0,\infty), \\ \tilde{u} &= u \quad \text{on } \Omega \times \{0\}, \\ \tilde{u} &= 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned}$$

- Unique continuation can be proved using the same Carleman estimate in $\mathbb{R}_{+}^{n+1}$.

THANK YOU