UNIQUE CONTINUATION FROM THE INFINITY FOR THE FRACTIONAL LAPLACIAN WITH A DRIFT TERM

PU-ZHAO KOW AND JENN-NAN WANG

Abstract. In this paper, we study a Landis-type conjecture for the fractional Schrödinger equation with a drift term. The Landis-type conjecture is a question of unique continuation property from the infinity. More precisely, we would like to prove the following: if any solution of the fractional Schrödinger equation with a drift term decays at a certain exponential rate, then such solution must be trivial. By localizing the equation with the help of Caffarelli-Silvestre extension, we solve the problem by a delicate Carleman inequality in the $(n+1)$-dimensional half space.

1. Introduction

In this work, we consider the unique continuation from the infinity for the fractional Schrödinger equation with drift term

$$((-\Delta)^s + b(x)x \cdot \nabla + q(x))u = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{1.1}$$

where $s \in (0,1)$ and b, q are scalar-valued functions. Precisely, we are interested in investigating the decay rate of u at infinity that implies the solution u is trivial. This problem is closely related to the Landis conjecture [KL88]. For $s=1$, $b=0$, Landis conjectured that if if $|q(x)| \le 1$ and $|u(x)| \le C_0$ satisfies $|u(x)| \le \exp(-C|x|^{1+})$, then $u \equiv 0$. The Landis conjecture was disproved by Meshkov [Me91], who constructed a complex-valued potential q and a nontrivial complex-valued u with $|u(x)| \le C\exp(-C|x|^{\frac{4}{3}})$ such that u is a solution of the Schrödinger equation with potential q. He also showed that if $|u(x)| \le C\exp(-C|x|^{\frac{4}{3}+})$, then $u \equiv 0$.

In view of Meshkov's counterexample, Kenig [Ke06] refined the Landis conjecture and asked whether this conjecture is true for real-valued potentials and solutions. This real version Landis conjecture was confirmed partially in [KSW15] where $n=2$ and $q \ge 0$. This result was later extended to the more general situation with Δ being replaced by any second order elliptic operator [DKW17]. The Landis conjecture in the real case with $n=1$ was studied in [Ro18]. Recently, the real version Landis conjecture in the plane case was resolved by Logunov, Malinnikova, Nadirashvili, Nazarov [LMNN20].

A Landis-type conjecture was considered in [RW19] for the fractional Laplacian with $b=0$. Both qualitative and quantitative estimates were proved in [RW19]. For example, when q is differentiable and satisfies

$$|x \cdot \nabla q(x)| \le 1,$$

if u satisfies the following decay behavior: $\exists\ \alpha > 1$ such that

$$\int_{\mathbb{R}^n} e^{|x|^\alpha} |u|^2 dx < \infty,$$

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then $u \equiv 0$. On the other hand, for a non-differentiable potential q, if $s \in (1/4, 1)$, $\|q\|_{L^\infty(\mathbb{R}^n)} \le 1$, and u satisfies the decay behavior: $\exists\, \alpha > \frac{4s}{4s-1}$ such that

$$\int_{\mathbb{R}^n} e^{|x|^\alpha} |u|^2 dx < \infty,$$

then $u \equiv 0$.

The main theme of this work is to extend the results in [RW19] to the fractional Schrödinger equation with a drift term (1.1). The unique continuation property established in [GSU20] states that if $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$ and $u = (-\Delta)^s u = 0$ in some open set $W \subset \mathbb{R}^n$, then $u \equiv 0$. It follows from this property that the unique continuation property holds for (1.1). However, to our best knowledge, the strong unique continuation property for the fractional Laplacian with a drift term remains open. Our work is the first attempt to study the unique continuation property for any solution of the fractional Laplacian with a drift term that satisfies some decaying condition. Motivated by the result of [Ro18], we consider the drift term only involving the radial derivative. Inspired by the ideas in [Me89] and [RW19], we show that if both b and q are differentiable, then any non-trivial solution of the fractional Schrödinger equation does not decay super-exponentially at infinity. The detailed statement is described in the following theorem.

Theorem 1.1. *Let $n = 1, 2, 3$ and*

$$\begin{cases} s \in (\frac{1}{2}, 1) & \text{when } n = 1 \text{ or } 2, \\ s \in (\frac{3}{4}, 1) & \text{when } n = 3. \end{cases} \tag{1.2}$$

We also assume that there exists a constant λ such that

$$|q(x)| \le \lambda \quad \text{and} \quad 0 \le b(x) \le \lambda |x|^{-\beta} \quad \text{for all } x \in \mathbb{R}^n \tag{1.3}$$

and the radial derivatives of q, b satisfy

$$|x \cdot \nabla q(x)| \le \lambda \quad \text{and} \quad |x \cdot \nabla b(x)| \le \lambda |x|^{-\beta} \quad \text{for all } x \in \mathbb{R}^n \tag{1.4}$$

for some $\beta > 1$. If $u \in H^s(\mathbb{R}^n)$ is a solution to (1.1) such that

$$\int_{\mathbb{R}^n} e^{|x|^\beta} \left[|u(x)|^2 + |\nabla u(x)|^2 \right] dx \le \lambda, \tag{1.5}$$

then $u \equiv 0$.

For non-differentiable potential q, we also can prove the following result.

Theorem 1.2. *Let $n = 1, 2, 3$ and s given in (1.2). Let $\beta > \frac{4s}{4s-1}$, and both b, q satisfy (1.3). Here we assume the radial derivative of b satisfies*

$$|x \cdot \nabla b(x)| \le \lambda |x|^{-\beta} \quad \text{for all } x \in \mathbb{R}^n.$$

If $u \in H^s(\mathbb{R}^n)$ is a solution to (1.1) satisfying (1.5), then $u \equiv 0$.

Remark 1.3. *Since we treat the drift term as a lower order addition, it is reasonable to expect that $s > 1/2$ in Theorem 1.1 and 1.2.*

Like several existing results, we want to prove Theorem 1.1 and 1.2 by an appropriate Carleman estimate. Since such estimate is a local estimate, we first localize the equation (1.1) by the Caffarelli-Silvestre extension [CS07] and derive the Carleman estimate for a degenerate elliptic equation in the $(n+1)$-dimensional upper half-space $\mathbb{R}^{n+1}_+$.

Our strategy in proving both theorems is similar to that of [RW19]. We will mainly handle the CS extended solution $\tilde{u}$ in $\mathbb{R}^{n+1}_+$. The condition (1.5) can be considered as a boundary decay for $\tilde{u}$. We then pass the boundary decay to the bulk decay of $\tilde{u}$ in $\mathbb{R}^{n+1}_+$. We would like to point out that unlike the pure potential case considered in [RW19], here, in order to guarantee the bulk decay of $\tilde{u}$, we also need the boundary decay of ∇u due to the addition of the drift term.

We now comment on the form of drift coefficient in (1.1). Such choice is due to the limitation in the Carleman estimate for $\tilde{u}$ in $\mathbb{R}^{n+1}_+$. Since the boundary term contains the first derivatives of u, to bound this boundary term, we need to include the second derivatives of $\tilde{u}$ in the Carleman estimate in view of the trace inequality (Lemma A.3). However, the parameter appears in the first derivatives of u is τ, while the parameter in the second derivatives of $\tilde{u}$ is τ^{-1} (see (4.4)). Without further restrictions, this boundary term can not be removed. We also remark that the operator

$$-\left((-\Delta)^s + \lambda x \cdot \nabla\right) \quad \text{(where } \lambda \text{ is a positive constant)} \tag{1.6}$$

is related to a $2s$-stable Ornstein-Uhlenbeck process in $\mathbb{R}^d$ for $s \in (0,1)$, see [Jak08, Sect. 2]. Precisely, let X_t be the $2s$-stable Ornstein-Uhlenbeck process in $\mathbb{R}^d$, given in the following stochastic integral:

$$X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)}\, d\hat{X}_s, \quad \hat{X}_0 = 0,$$

where the integral is in the Stieltjes sense, and $(\hat{X}_t, \mathbb{P}^x)$ is the isotropic $2s$-stable Lévy process in $\mathbb{R}^d$ with index of stability $s \in (0,1)$ and characteristic function

$$\mathbb{E}^0 e^{i\hat{X}_t \cdot \xi} = e^{-t|\xi|^{2s}} \quad \text{for all } \xi \in \mathbb{R}^d \text{ and for all } t \geq 0.$$

Here, $\mathbb{E}^x$ denotes the expectation with respect to the distribution $\mathbb{P}^x$ of the process starting from $x \in \mathbb{R}^d$. Indeed, the infinitesimal genrator of X_t is equal to (1.6).

This paper is organized as follows. In Section 2, we introduce some notations and state the Caffarelli-Silvestre extension. In Section 3, we discuss how the boundary decay of u implies the bulk decay of $\tilde{u}$ in $\mathbb{R}^{n+1}_+$. The derivation of the needed Carleman estimate is discussed in great detail in Section 4. Section 5 is devoted to the proofs of main results.

2. The Caffarelli-Silvestre extension

Basically, we shall follow the definitions and notations in [RW19]. We now restate what we need. Let $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \mathbb{R}_+ = \left\{ x = (x', x_{n+1}) \mid x' \in \mathbb{R}^n, x_{n+1} > 0 \right\}$ and $x_0 = (x', 0) \in \mathbb{R}^n \times \{0\}$. For $r, R > 0$, we denote

$$\begin{aligned}
B_r^+(x_0) &:= \left\{ x \in \mathbb{R}^{n+1}_+ \mid |x - x_0| \leq r \right\}, \\
B_r'(x_0) &:= \left\{ x \in \mathbb{R}^n \times \{0\} \mid |x - x_0| \leq r \right\}, \\
B_r^+ &:= B_r^+(0), \quad B_r' := B_r'(0), \\
A_{r,R}^+ &:= \left\{ x \in \mathbb{R}^{n+1}_+ \mid r \leq |x| \leq R \right\}, \\
A_{r,R}' &:= \left\{ x \in \mathbb{R}^n \times \{0\} \mid r \leq |x| \leq R \right\}.
\end{aligned}$$

For $\Omega \subset \mathbb{R}^{n+1}_+$, we define

$$\dot{H}^1(\Omega, x_{n+1}^{1-2s}) := \left\{ \ v : \Omega \to \mathbb{R} \ \middle| \ \int_\Omega x_{n+1}^{1-2s} |\nabla v|^2 \, dx < \infty \ \right\},$$
$$H^1(\Omega, x_{n+1}^{1-2s}) := \left\{ \ v : \Omega \to \mathbb{R} \ \middle| \ \int_\Omega x_{n+1}^{1-2s} (|v|^2 + |\nabla v|^2) \, dx < \infty \ \right\}.$$

Write $x = r\theta$, where $r > 0$ and $\theta \in \mathcal{S}^n_+$, i.e., $\theta = (\theta', \theta_{n+1}) \in \mathcal{S}^n$ with $\theta_{n+1} > 0$. We also denote

$$H^1(\mathcal{S}^n_+, \theta_{n+1}^{1-2s}) := \left\{ \ v : \mathcal{S}^n_+ \to \mathbb{R} \ \middle| \ \int_{\mathcal{S}^n_+} \theta_{n+1}^{1-2s} (|v|^2 + |\nabla_{\mathcal{S}^n} v|^2) \, d\theta < \infty \ \right\},$$

where $\nabla_{\mathcal{S}^n} = (\Theta_1, \cdots, \Theta_{n+1})$. Here, Θ_k are vector fields on $\mathcal{S}^n$.

For $s \in (\frac{1}{2}, 1)$ and $u \in H^s(\mathbb{R}^n)$, let $\tilde{u} \in \dot{H}^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ be a solution to the degenerate elliptic equation

$$\begin{cases} \nabla \cdot x_{n+1}^{1-2s} \nabla \tilde{u} = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on} \quad \mathbb{R}^n \times \{0\}, \end{cases} \tag{2.1}$$

where $\nabla = (\nabla', \partial_{n+1}) = (\partial_1, \cdots, \partial_n, \partial_{n+1})$. It was established in [CS07] that

$$-(-\Delta)^s u(x) = c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(x)$$

for some constant $c_{n,s} > 0$. In view of this observation, (1.1) can be reformulated as the local, degenerate elliptic equation

$$\begin{cases} \nabla \cdot x_{n+1}^{1-2s} \nabla \tilde{u} = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ \tilde{u} = u & \text{on} \quad \mathbb{R}^n \times \{0\}, \\ c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} = bx' \cdot \nabla' u + qu & \text{on} \quad \mathbb{R}^n \times \{0\}. \end{cases} \tag{2.2}$$

3. Boundary decay implies bulk decay

In this section, we will show that the boundary decay (1.5) implies the bulk decay of $\tilde{u}$ in $\mathbb{R}^{n+1}_+$.

Proposition 3.1. *Let s be given in (1.2) and $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ be a solution to (2.2). Assume that there exists a constant $\lambda > 0$ such that*

$$|q(x')| \le \lambda \quad \textit{and} \quad |b(x')||x'| \le \lambda \quad \textit{for all} \quad x' \in \mathbb{R}^n. \tag{3.1}$$

If there exist constants $C, \beta > 1$ such that (1.5) holds, then there exist constants $C_1, c_1 > 0$ such that

$$|\tilde{u}(x)| \le C_1 e^{-c_1 |x|^\beta} \quad \textit{for all} \quad x \in \mathbb{R}^{n+1}_+. \tag{3.2}$$

The following lemma can be found in [RW19, Equ.(19)].

Lemma 3.2. *Let $s \in (\frac{1}{2}, 1)$ and $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ be a solution to (2.1). If $x_0 \in \mathbb{R}^n \times \{0\}$, then there exist $\alpha = \alpha(n,s) \in (0,1)$ and $c = c(n,s) \in (0,1)$ such that*

$$
\begin{aligned}
&\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B^+_{cr}(x_0))} \\
&\leq C \left[\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B^+_{16r}(x_0))} + r^{1-s} \|u\|_{L^2(B'_{16r}(x_0))} \right]^{\alpha} \\
&\qquad \times \left[r^{s+1} \left\| \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} \right\|_{L^2(B'_{16r}(x_0))} + r^{1-s} \|u\|_{L^2(B'_{16r}(x_0))} \right]^{1-\alpha}
\end{aligned}
\tag{3.3}
$$

for some positive constant C.

Combining Lemma 3.2 and Lemma A.2 implies the following lemma.

Lemma 3.3. *Let s be given by (1.2) and $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ be a solution to (2.2). If $q \in L^\infty(\mathbb{R}^n)$ and $b(x')x' \in (L^\infty(\mathbb{R}^n))^n$, then there exist $\alpha = \alpha(n,s) \in (0,1)$ and $c = c(n,s) \in (0,1)$ such that*

$$
\begin{aligned}
&\|\tilde{u}\|_{L^\infty(B^+_{\frac{cr}{4}})} \\
&\leq C r^{-\frac{n}{2}} \Bigg[\left(r^{s-1} \|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B^+_{16r})} + \|u\|_{L^2(B'_{16r})} \right)^{\alpha} \\
&\qquad \times \left(r^{2s} \|bx' \cdot \nabla' u + qu\|_{L^2(B'_{16r})} + \|u\|_{L^2(B'_{16r})} \right)^{1-\alpha} \\
&\qquad + r^{2s} \|bx' \cdot \nabla' u + qu\|_{L^2(B'_{16r})} \Bigg]
\end{aligned}
\tag{3.4}
$$

for all $r > 1$, where the constant C is independent to r.

Proof. Given any $r > 1$, let $\tilde{v}(x) = \tilde{u}(rx)$ in $\mathbb{R}^{n+1}_+$ and $v(x') = u(rx')$ on $\mathbb{R}^n \times \{0\}$. Note that

$$
\begin{aligned}
c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{v}(x) &= c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \left[\tilde{u}(rx) \right] \\
&= r c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(rx) \\
&= r c_{n,s} \lim_{r x_{n+1} \to 0} r^{-1+2s} (r x_{n+1})^{1-2s} \partial_{n+1} \tilde{u}(rx) \\
&= r^{2s} c_{n,s} \lim_{r x_{n+1} \to 0} (r x_{n+1})^{1-2s} \partial_{n+1} \tilde{u}(rx) \\
&= r^{2s} \left[b(rx') rx' \cdot \nabla' u(rx) + q(rx) u(rx) \right] \quad \text{(By (2.2))} \\
&= r^{2s} b(rx') x' \cdot \nabla' \left[u(rx) \right] + r^{2s} q(rx) u(rx) \\
&= r^{2s} b(rx') x' \cdot \nabla' v(x) + r^{2s} q(rx) v(x),
\end{aligned}
$$

that is,

$$
\begin{cases}
\nabla \cdot x_{n+1}^{1-2s} \nabla \tilde{v} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\tilde{v} = v & \text{on } \mathbb{R}^n \times \{0\}, \\
c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{v} = b_r x' \cdot \nabla' v + q_r v & \text{on } \mathbb{R}^n \times \{0\},
\end{cases}
\tag{3.5}
$$

where $b_r(x') := r^{2s}b(rx')$ and $q_r(x') := r^{2s}q(rx)$. Applying Lemma A.2 to (3.5) with $p = 2$ (since (1.2)) $a_1 = 0$ and $a_2 = b_r x' \cdot \nabla' v + q_r v$, we obtain that

$$\|\tilde{v}\|_{L^\infty(B^+_{1/4})} \le C\left[\|x_{n+1}^{\frac{1-2s}{2}}\tilde{v}\|_{L^2(B_1^+)} + \|b_r x' \cdot \nabla' v + q_r v\|_{L^2(B_1')}\right], \tag{3.6}$$

where $C = C(n, s)$. Since $\tilde{v}(x) = \tilde{u}(rx)$, we have

$$\|\tilde{v}\|_{L^\infty(B^+_{1/4})} = \sup_{x\in\mathbb{R}^{n+1}_+, |x|\le\frac{1}{4}} |\tilde{v}(x)| = \sup_{x\in\mathbb{R}^{n+1}_+, |rx|\le\frac{r}{4}} |\tilde{u}(rx)| = \|\tilde{u}\|_{L^\infty(B^+_{r/4})}.$$

On the other hand, we can derive

$$\|x_{n+1}^{\frac{1-2s}{2}}\tilde{v}\|^2_{L^2(B_1^+)} = \int_{x\in\mathbb{R}^{n+1}_+, |x|\le 1} x_{n+1}^{1-2s}|\tilde{v}(x)|^2\,dx = r^{2s-2-n}\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|^2_{L^2(B_r^+)}$$

and

$$\begin{aligned}\|b_r x' \cdot \nabla' v + q_r v\|^2_{L^2(B_1')} &= \int_{|x'|<1} |b_r(x')x' \cdot \nabla' v(x') + q_r(x')v(x')|^2\,dx' \\ &= r^{4s-n}\|bx' \cdot \nabla' u + qu\|^2_{L^2(B_r')}.\end{aligned}$$

Thus, (3.6) implies

$$\|\tilde{u}\|_{L^\infty(B^+_{r/4})} \le Cr^{-\frac{n}{2}}\left[r^{s-1}\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B_r^+)} + r^{2s}\|bx' \cdot \nabla' u + qu\|_{L^2(B_r')}\right].$$

Here, $r > 1$ is arbitrary and C is independent of r. Replacing r by cr, where c is the constant given in Lemma 3.2, we have

$$\|\tilde{u}\|_{L^\infty(B^+_{cr/4})} \le Cr^{-\frac{n}{2}}\left[r^{s-1}\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B_{cr}^+)} + r^{2s}\|bx' \cdot \nabla' u + qu\|_{L^2(B_{cr}')}\right].$$

Here, C is another constant independent of r. Combining this inequality with (3.3) and using

$$c_{n,s}\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u} = bx' \cdot \nabla' u + qu,$$

we obtain our desired result. □

We are ready to prove the main result of this section.

Proof of Proposition 3.1. Let $R > 1$ be sufficiently large and $x_0 \in \mathbb{R}^n \times \{0\}$ with $|x_0| = 32R$. From (1.5), it follows that

$$\begin{aligned}\lambda &\ge \int_{B'_{16R}(x_0)} e^{|x'|^\beta}\left[|u(x')|^2 + |\nabla' u(x')|^2\right]dx' \\ &\ge e^{(16R)^\beta}\int_{B'_{16R}(x_0)}\left[|u(x')|^2 + |\nabla' u(x')|^2\right]dx' \\ &= e^{(16R)^\beta}\left[\|u\|^2_{L^2(B'_{16R}(x_0))} + \|\nabla' u\|^2_{L^2(B'_{16R}(x_0))}\right],\end{aligned}$$

that is,

$$\|u\|_{L^2(B'_{16R}(x_0))} + \|\nabla' u\|_{L^2(B'_{16R}(x_0))} \le \tilde{C}e^{-\tilde{c}R^\beta} \tag{3.7}$$

for some constants $\tilde{C}$ and $\tilde{c}$. Note that the Caffarelli-Silvestre extension $\tilde{u}$ satisfies

$$\|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)} \le C \quad\text{and}\quad \|\tilde{u}(\bullet, x_{n+1})\|_{L^2(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^n)}$$

(see [St10, page 48-49]). Thus, we have

$$\begin{aligned}\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|^2_{L^2(B^+_{16R}(x_0))} &\leq \int_0^{16R} x_{n+1}^{1-2s}\|\tilde{u}(\bullet,x_{n+1})\|^2_{L^2(\mathbb{R}^n)}\,dx_{n+1}\\ &\leq \int_0^{16R} x_{n+1}^{1-2s}\|u\|^2_{L^2(\mathbb{R}^n)}\,dx_{n+1}\\ &= \frac{(16R)^{2-2s}}{2-2s}\|u\|^2_{L^2(\mathbb{R}^n)}\\ &\leq CR^{2-2s}.\end{aligned}$$

Plugging $c_{n,s}\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u} = bx'\cdot\nabla' u + qu$ into (3.3) (with $r = R$) and using (3.1) yields

$$\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B^+_{cR}(x_0))} \leq \tilde{C}e^{-\tilde{c}R^\beta}.$$

Next, choosing $r = \frac{cR}{16}$ in (3.4) and putting together (3.1) and (1.5), we have

$$\|\tilde{u}\|_{L^\infty(B^+_{\frac{c^2R}{64}}(x_0))} \leq \tilde{C}e^{-\tilde{c}R^\beta} \quad \text{for} \quad |x_0| = 32R,$$

which implies

$$\|\tilde{u}\|_{L^\infty(B^+_{c\tilde{R}}(x_0))} \leq \tilde{C}e^{-\tilde{c}\tilde{R}^\beta} \quad \text{for all large} \quad \tilde{R} \gg 1.$$

The decay estimate (3.2) then follows from the chain-of-balls argument described in [RW19, Proposition 2.2, Step 2]. □

4. Carleman estimates

This section is mainly devoted to the derivations of Carleman estimates. We will discuss the estimates corresponding to both differentiable and non-differentiable potentials.

4.1. Carleman estimate with differentiable potential.

The proof of the Carleman estimate below follows from the argument in [RW19].

Theorem 4.1. *Let $s \in (\frac{1}{2}, 1)$ and let $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ with $\operatorname{supp}(\tilde{u}) \subset \mathbb{R}^{n+1}_+ \setminus B_1^+$ be a solution to*

$$\begin{cases}\nabla\cdot x_{n+1}^{1-2s}\nabla\tilde{u} = f & \text{in} \quad \mathbb{R}^{n+1}_+,\\ \tilde{u} = u & \text{on} \quad \mathbb{R}^n\times\{0\},\\ \lim\limits_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u} = Wx'\cdot\nabla' u + Vu + g & \text{on} \quad \mathbb{R}^n\times\{0\},\end{cases}$$

where $f \in L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{2s-1})$ is compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, $g \in L^2(\mathbb{R}^n)$ is compactly supported in $\mathbb{R}^n$, and $x'\cdot\nabla' W$, $x'\cdot\nabla' V$ exist. Let $\alpha > 1$ be a real constant and define $\phi(x) := |x|^\alpha$. If

$$W(x') \geq 0 \quad \text{for all} \quad x' \in \mathbb{R}^n \setminus B_1'.$$

Then there exists a real number $\tau_0 > 1$ such that

$$
\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\alpha}{2}-1}x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\
&+ \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\alpha}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla(x\cdot\nabla\tilde{u})\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\
\le\ & C\Big[\|e^{\tau\phi}|x|x_{n+1}^{\frac{2s-1}{2}}f\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\
&+ \tau\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}|V|^{\frac{1}{2}}u\|^2_{L^2(\mathbb{R}^n\times\{0\})} + \tau\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}|x'\cdot\nabla'V|^{\frac{1}{2}}u\|^2_{L^2(\mathbb{R}^n\times\{0\})} \\
&+ \tau^2\|e^{\tau\phi}|x|^{\alpha}|W|^{\frac{1}{2}}\tilde{u}\|^2_{L^2(\mathbb{R}^n\times\{0\})} + \tau^2\|e^{\tau\phi}|x|^{\alpha}|x'\cdot\nabla'W|^{\frac{1}{2}}u\|^2_{L^2(\mathbb{R}^n\times\{0\})} \\
&+ \tau^2\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}u\|^2_{L^2(\mathbb{R}^n\times\{0\})} \\
&+ \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\alpha}{2}}|V|^{\frac{1}{2}}(x'\cdot\nabla'u)\|^2_{L^2(\mathbb{R}^n\times\{0\})} \\
&+ \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\alpha}{2}}|x'\cdot\nabla'V|^{\frac{1}{2}}(x'\cdot\nabla'u)\|^2_{L^2(\mathbb{R}^n\times\{0\})} \\
&+ \|e^{\tau\phi}|W|^{\frac{1}{2}}(x'\cdot\nabla'u)\|^2_{L^2(\mathbb{R}^n\times\{0\})} + \|e^{\tau\phi}|x'\cdot\nabla'W|^{\frac{1}{2}}(x'\cdot\nabla'u)\|^2_{L^2(\mathbb{R}^n\times\{0\})} \\
&+ \|e^{\tau\phi}|x|^{-\frac{\alpha}{2}}(x'\cdot\nabla'u)\|^2_{L^2(\mathbb{R}^n\times\{0\})} \\
&+ \tau^2\|e^{\tau\phi}|x|^{\frac{3}{2}\alpha}|x|^{\frac{1+2s}{2}}g\|^2_{L^2(\mathbb{R}^n\times\{0\})} + \|e^{\tau\phi}|x|^{-\frac{\alpha}{2}}|x|^{s}(x'\cdot\nabla'g)\|^2_{L^2(\mathbb{R}^n\times\{0\})}\Big]
\end{aligned}
\tag{4.1}
$$

for all $\tau \ge \tau_0$. Here, the positive constant C is independent of τ.

Proof. **Step 1: Pass to conformal polar coordinates.** Let $x = e^t\theta$ with $t \in \mathbb{R}$ and $\theta \in \mathcal{S}^n_+$. Set $\overline{u} := e^{\frac{n-2s}{2}t}\tilde{u}$, we have

$$
\begin{cases}
\left[\theta_{n+1}^{1-2s}\partial_t^2 + \nabla_{\mathcal{S}^n}\cdot\theta_{n+1}^{1-2s}\nabla_{\mathcal{S}^n} - \theta_{n+1}^{1-2s}\dfrac{(n-2s)^2}{4}\right]\overline{u} = \tilde{f} & \text{in } \ \mathcal{S}^n_+\times\mathbb{R}, \\[2ex]
\lim\limits_{\theta_{n+1}\to 0}\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\overline{u} = \left(\tilde{V} - \dfrac{n-2s}{2}\tilde{W}\right)\overline{u} + \tilde{W}\partial_t\overline{u} + \tilde{g} & \text{on } \ \partial\mathcal{S}^n_+\times\mathbb{R},
\end{cases}
\tag{4.2}
$$

where $\tilde{f} = e^{\frac{n+2+2s}{2}t}f$, $\nu = (0,\cdots,0,1)$, $\tilde{V} = e^{2st}V$, $\tilde{W} = e^{2st}W$, and $\tilde{g} = e^{\frac{n+2s}{2}t}g$.

Next, by setting $\tilde{\Delta}_{\mathcal{S}^n} := \theta_{n+1}^{\frac{2s-1}{2}}\nabla_{\mathcal{S}^n}\cdot\theta_{n+1}^{1-2s}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}$, $\varphi(t) = \phi(e^t\theta) = e^{\alpha t}$, $\overline{v} = e^{\tau\varphi}\theta_{n+1}^{\frac{1-2s}{2}}\overline{u}$, $\overline{f} = e^{\tau\varphi}\tilde{f}$, and $\overline{g} = e^{\tau\varphi}\tilde{g}$, (4.2) can be written as

$$
\begin{cases}
L_\varphi\overline{v} := (S+A)\overline{v} = \theta_{n+1}^{\frac{2s-1}{2}}\overline{f} & \text{in } \ \mathcal{S}^n_+\times\mathbb{R}, \\[2ex]
\lim\limits_{\theta_{n+1}\to 0}\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v} = I + \tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v} + \overline{g} & \text{on } \ \partial\mathcal{S}^n_+\times\mathbb{R},
\end{cases}
$$

where

$$
\begin{aligned}
S &:= \partial_t^2 + \tilde{\Delta}_{\mathcal{S}^n} - \frac{(n-2s)^2}{4} + \tau^2|\varphi'|^2, \\
A &:= -2\tau\varphi'\partial_t - \tau\varphi'', \\
I &:= \left(\tilde{V} - \frac{n-2s}{2}\tilde{W}\right)\theta_{n+1}^{\frac{2s-1}{2}}\overline{v} - \tau\varphi'\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}.
\end{aligned}
$$

To shortened the notations, we denote the norm and the scalar product in the bulk and the boundary space by

$$\begin{cases} \| \bullet \| := \| \bullet \|_{L^2(\mathcal{S}^n_+ \times \mathbb{R})}, & \langle \bullet, \bullet \rangle := \langle \bullet, \bullet \rangle_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} \\ \| \bullet \|_0 := \| \bullet \|_{L^2(\partial \mathcal{S}^n_+ \times \mathbb{R})}, & \langle \bullet, \bullet \rangle_0 := \langle \bullet, \bullet \rangle_{L^2(\partial \mathcal{S}^n_+ \times \mathbb{R})} \end{cases} \tag{4.3}$$

and we omit the notation "$\lim_{\theta_{n+1} \to 0}$" in $\| \bullet \|_0$ and $\langle \bullet, \bullet \rangle_0$.

Since S, A are only symmetric and anti-symmetric up to boundary contributions, we can see that

$$\begin{aligned} \|L_\varphi \overline{v}\|^2 = \|S\overline{v}\|^2 + \|A\overline{v}\|^2 + \langle [S,A]\overline{v}, \overline{v} \rangle \\ + 4\tau \langle \theta_{n+1}^{1-2s} \nu \cdot \nabla_{\mathcal{S}^n} \theta_{n+1}^{\frac{2s-1}{2}} \overline{v}, \varphi' \theta_{n+1}^{\frac{2s-1}{2}} \partial_t \overline{v} \rangle_0 \\ + 2\tau \langle \theta_{n+1}^{1-2s} \nu \cdot \nabla_{\mathcal{S}^n} \theta_{n+1}^{\frac{2s-1}{2}} \overline{v}, \varphi'' \theta_{n+1}^{\frac{2s-1}{2}} \overline{v} \rangle_0. \end{aligned} \tag{4.4}$$

From $\tilde{W} \geq 0$, it follows that $\langle \tilde{W} \theta_{n+1}^{\frac{2s-1}{2}} \partial_t \overline{v}, \varphi' \theta_{n+1}^{\frac{2s-1}{2}} \partial_t \overline{v} \rangle_0 \geq 0$ and

$$\begin{aligned} \|L_\varphi \overline{v}\|^2 \geq \|S\overline{v}\|^2 + \|A\overline{v}\|^2 + \langle [S,A]\overline{v}, \overline{v} \rangle \\ + 4\tau \langle I, \varphi' \theta_{n+1}^{\frac{2s-1}{2}} \partial_t \overline{v} \rangle_0 + 4\tau \langle \overline{g}, \varphi' \theta_{n+1}^{\frac{2s-1}{2}} \partial_t \overline{v} \rangle_0 \\ + 2\tau \langle \theta_{n+1}^{1-2s} \nu \cdot \nabla_{\mathcal{S}^n} \theta_{n+1}^{\frac{2s-1}{2}} \overline{v}, \varphi'' \theta_{n+1}^{\frac{2s-1}{2}} \overline{v} \rangle_0. \end{aligned} \tag{4.5}$$

Step 2: Estimate the commutator term $\langle [S,A]\overline{v}, \overline{v} \rangle$. The commutator term can be computed explicitly

$$[S,A]\overline{v} = 4\tau^3 |\varphi'|^2 \varphi'' \overline{v} - 4\tau \varphi'' \partial_t^2 \overline{v} - 4\tau \varphi''' \partial_t \overline{v} - \tau \varphi'''' \overline{v}.$$

Since $\mathrm{supp}(\tilde{u}) \subset \mathbb{R}^{n+1}_+ \setminus B_1^+$, we can see that $\mathrm{supp}(\overline{v}) \subset \left\{ (\theta, t) \in \mathcal{S}^n_+ \times \mathbb{R} \,\middle|\, t \geq 0 \right\}$. Recall that $\varphi(t) = e^{\alpha t}$ for $\alpha > 1$. We thus have $|\varphi'|^2 \varphi'' \geq \varphi''''$ in $\mathrm{supp}(\overline{v})$. By this inequality, we can estimate

$$\langle [S,A]\overline{v}, \overline{v} \rangle \geq 2\tau^3 \|\varphi' (\varphi'')^{\frac{1}{2}} \overline{v}\|^2 + 4\tau \|(\varphi'')^{\frac{1}{2}} \partial_t \overline{v}\|^2.$$

Combining this inequality with (4.5) yields

$$\begin{aligned} \|L_\varphi \overline{v}\|^2 \geq \|S\overline{v}\|^2 + \|A\overline{v}\|^2 + 2\tau^3 \|\varphi' (\varphi'')^{\frac{1}{2}} \overline{v}\|^2 + 4\tau \|(\varphi'')^{\frac{1}{2}} \partial_t \overline{v}\|^2 \\ + 4\tau \langle I, \varphi' \theta_{n+1}^{\frac{2s-1}{2}} \partial_t \overline{v} \rangle_0 + 4\tau \langle \overline{g}, \varphi' \theta_{n+1}^{\frac{2s-1}{2}} \partial_t \overline{v} \rangle_0 \\ + 2\tau \langle \theta_{n+1}^{1-2s} \nu \cdot \nabla_{\mathcal{S}^n} \theta_{n+1}^{\frac{2s-1}{2}} \overline{v}, \varphi'' \theta_{n+1}^{\frac{2s-1}{2}} \overline{v} \rangle_0. \end{aligned} \tag{4.6}$$

Step 3: Derive the full gradient inequality from the symmetric part $\|S\overline{v}\|^2$. Let c_0 be a positive constant to be determined later. Note that

$$-\tilde{\Delta}_{\mathcal{S}^n} = -S + \partial_t^2 + \tau^2 (\varphi')^2 - \frac{(n-2s)^2}{4}.$$

Hence, we have

$$\begin{aligned}
&c_0\tau\|(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2 + c_0\tau\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0\\
&= -c_0\tau\langle\varphi'\overline{v},\tilde{\Delta}_{\mathcal{S}^n}\overline{v}\rangle\\
&= c_0\tau\Big[-\langle\varphi'\overline{v},S\overline{v}\rangle - \langle\varphi''\overline{v},\partial_t\overline{v}\rangle + \|(\varphi')^{\frac{1}{2}}\partial_t\overline{v}\|^2 + \tau^2\|(\varphi')^{\frac{3}{2}}\overline{v}\|^2\\
&\qquad - \frac{(n-2s)^2}{4}\|(\varphi')^{\frac{1}{2}}\overline{v}\|^2\Big]\\
&\le -c_0\tau\langle\varphi'\overline{v},S\overline{v}\rangle - c_0\tau\langle\varphi''\overline{v},\partial_t\overline{v}\rangle + c_0\tau\|(\varphi')^{\frac{1}{2}}\partial_t\overline{v}\|^2 + 2c_0\tau^3\|(\varphi')^{\frac{3}{2}}\overline{v}\|^2 \qquad (4.7)
\end{aligned}$$

for all $\tau \gg 1$. Combining (4.6) and (4.7) gives

$$\begin{aligned}
&\|S\overline{v}\|^2 + \|A\overline{v}\|^2 + 2\tau^3\|\varphi'(\varphi'')^{\frac{1}{2}}\overline{v}\|^2 + 4\tau\|(\varphi'')^{\frac{1}{2}}\partial_t\overline{v}\|^2\\
&+ c_0\tau\|(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2\\
&+ 4\tau\langle I,\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0 + 4\tau\langle\overline{g},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0\\
&+ 2\tau\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0\\
&+ c_0\tau\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0\\
&\le \|L_\varphi\overline{v}\|^2 - c_0\tau\langle\varphi'\overline{v},S\overline{v}\rangle - c_0\tau\langle\varphi''\overline{v},\partial_t\overline{v}\rangle\\
&+ c_0\tau\|(\varphi')^{\frac{1}{2}}\partial_t\overline{v}\|^2 + 2c_0\tau^3\|(\varphi')^{\frac{3}{2}}\overline{v}\|^2.
\end{aligned}$$

Using $\varphi'' = \alpha\varphi'$ and choosing sufficiently small $c_0 > 0$, we can derive

$$\begin{aligned}
&\|S\overline{v}\|^2 + \|A\overline{v}\|^2 + \tau^3\|\varphi'(\varphi'')^{\frac{1}{2}}\overline{v}\|^2\\
&+ \tau\|(\varphi'')^{\frac{1}{2}}\partial_t\overline{v}\|^2 + \tau\|(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2\\
&\le C\Big[\|L_\varphi\overline{v}\|^2 + \tau|\langle I,\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| + \tau|\langle\overline{g},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|\\
&\qquad + \tau|\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0|\Big] \qquad (4.8)
\end{aligned}$$

for some positive constant C, which is independent of τ.

Step 4: Estimate the second derivatives from the symmetric part $\|S\overline{v}\|^2$. Let c_1 be a positive constant to be determined later. In view of $\varphi' \ge 1$ in $\mathrm{supp}(\overline{v})$, we obtain

$$\|S\overline{v}\|^2 \ge c_1\tau^{-1}\|(\varphi')^{-\frac{1}{2}}S\overline{v}\|^2 \ge c_1\tau^{-1}\|(\varphi')^{-\frac{1}{2}}S'\overline{v}\|^2 - C\tau^{-1}\|(\varphi')^{-\frac{1}{2}}\overline{v}\|^2,$$

where $S' := \partial_t^2 + \tilde{\Delta}_{\mathcal{S}^n} + \tau^2(\varphi')^2$. It is clear that

$$\begin{aligned}
\|(\varphi')^{-\frac{1}{2}}S'\overline{v}\|^2 &= \|(\varphi')^{-\frac{1}{2}}\partial_t^2\overline{v}\|^2 + \|(\varphi')^{-\frac{1}{2}}\tilde{\Delta}_{\mathcal{S}^n}\overline{v}\|^2 + \tau^4\|(\varphi')^{\frac{3}{2}}\overline{v}\|^2\\
&\quad + 2\langle(\varphi')^{-1}\partial_t^2\overline{v},\tilde{\Delta}_{\mathcal{S}^n}\overline{v}\rangle + 2\tau^2\langle\partial_t^2\overline{v},\varphi'\overline{v}\rangle + 2\tau^2\langle\tilde{\Delta}_{\mathcal{S}^n}\overline{v},\varphi'\overline{v}\rangle.
\end{aligned}$$

Repeating the integration by parts gives

$$\langle\partial_t^2\overline{v},\varphi'\overline{v}\rangle = -\langle\partial_t\overline{v},\varphi''\overline{v}\rangle - \|(\varphi')^{\frac{1}{2}}\partial_t\overline{v}\|^2,$$

$$\langle\tilde{\Delta}_{\mathcal{S}^n}\overline{v},\varphi'\overline{v}\rangle = -\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0 - \|(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2,$$

and

$$\begin{aligned}\langle (\varphi')^{-1}\partial_t^2\overline{v}, \tilde{\Delta}_{\mathcal{S}^n}\overline{v}\rangle =& -\langle \theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0\\ &-\frac{1}{2}\|\varphi^{-1}(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2\\ &+\|(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|^2.\end{aligned}$$

Therefore, we can derive

$$\begin{aligned}\|S\overline{v}\|^2 \geq\,& c_1\tau^{-1}\|(\varphi')^{-\frac{1}{2}}S\overline{v}\|^2\\ \geq\,& c_1\tau^{-1}\|(\varphi')^{-\frac{1}{2}}\partial_t^2\overline{v}\|^2 + 2c_1\tau^{-1}\|(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|^2\\ &+c_1\tau^3\|(\varphi')^{\frac{3}{2}}\overline{v}\|^2 - c_1\tau^{-1}\|\varphi^{-1}(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2\\ &-c_1\tau\langle\partial_t\overline{v},\varphi''\overline{v}\rangle - c_1\tau\|(\varphi')^{\frac{1}{2}}\partial_t\overline{v}\|^2\\ &-2c_1\tau^{-1}\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0\\ &-2c_1\tau\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi'\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0. \end{aligned}\tag{4.9}$$

Putting this inequality and (4.8) together and choosing sufficiently small $c_1 > 0$, we have

$$\begin{aligned}&\tau^3\|\varphi'(\varphi'')^{\frac{1}{2}}\overline{v}\|^2 + \tau\|(\varphi'')^{\frac{1}{2}}\partial_t\overline{v}\|^2 + \tau\|(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2\\ &+\tau^{-1}\|(\varphi')^{-\frac{1}{2}}\partial_t^2\overline{v}\|^2 + \tau^{-1}\|(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|^2\\ \leq\,& C\Big[\|L_\varphi\overline{v}\|^2 + \tau|\langle I, \varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| + \tau|\langle \overline{g}, \varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|\\ &+\tau|\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0|\\ &+\tau^{-1}|\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0|\Big].\end{aligned}\tag{4.10}$$

Step 5.1: Estimate the term $\tau|\langle I, \varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| + \tau|\langle \overline{g}, \varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|$**.** Write $\tilde{U} := \tilde{V} - \frac{n-2s}{2}\tilde{W}$. Performing integration by parts leads to

$$\begin{aligned}&|\tau\langle\tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|\\ &=\tau\Big|-\frac{1}{2}\langle\tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0 - \frac{1}{2}\langle\partial_t\tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi'\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0\Big|\\ &\leq\frac{\tau}{2}\||\tilde{U}|^{\frac{1}{2}}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \frac{\tau}{2}\||\partial_t\tilde{U}|^{\frac{1}{2}}|\varphi'|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\ &\leq C\Big[\tau\||\tilde{V}|^{\frac{1}{2}}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau\||\partial_t\tilde{V}|^{\frac{1}{2}}|\varphi'|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\ &\qquad+\tau\||\tilde{W}|^{\frac{1}{2}}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau\||\partial_t\tilde{W}|^{\frac{1}{2}}|\varphi'|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\Big]\end{aligned}$$

and

$$\begin{aligned}
&|\tau\langle\tau\varphi'\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|\\
&=\tau^2|\langle(\varphi')^2\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|\\
&=\tau^2\left|-\frac{1}{2}\langle 2\varphi'\varphi''\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0-\frac{1}{2}\langle(\varphi')^2\partial_t\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0\right|\\
&\le\tau^2\|(\varphi')^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}|\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2+\frac{\tau^2}{2}\|\varphi'|\partial_t\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2.
\end{aligned}$$

Also, we have

$$\tau|\langle\overline{g},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|=|\langle\tau(\varphi')^{\frac{3}{2}}\overline{g},(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|\le\tau^2\|(\varphi')^{\frac{3}{2}}\overline{g}\|_0^2+\|(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2.$$

Combining all estimates together yields

$$\begin{aligned}
&\tau|\langle I,\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|+\tau|\langle\overline{g},\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0|\\
&\le C\Big[\tau\||\tilde{V}|^{\frac{1}{2}}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2+\tau\||\partial_t\tilde{V}|^{\frac{1}{2}}|\varphi'|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\
&\qquad+\tau^2\|(\varphi')^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}|\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2+\tau^2\|\varphi'|\partial_t\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\
&\qquad+\|(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2+\tau^2\|(\varphi')^{\frac{3}{2}}\overline{g}\|_0^2\Big].
\end{aligned}\tag{4.11}$$

Step 5.2: Estimate the term $\tau|\langle\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0|$. Recall that

$$\lim_{\theta_{n+1}\to 0}\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}=I+\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}+\overline{g}\quad\text{on}\quad\partial\mathcal{S}_+^n\times\mathbb{R}.$$

Similar as above, we can estimate

$$\begin{aligned}
&\tau|\langle I,\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0|\\
&\le C\Big[\tau\||\tilde{V}|^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2+\tau\||\tilde{W}|^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\
&\qquad+\tau^2\|(\varphi')^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}|\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\Big].
\end{aligned}$$

By using integration by parts, we have

$$\begin{aligned}
&\tau|\langle\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v},\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0|\\
&=\tau\left|-\frac{1}{2}\langle\partial_t\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0-\frac{1}{2}\langle\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v},\varphi'''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0\right|\\
&\le\tau\||\partial_t\tilde{W}|^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2+\tau\||\tilde{W}|^{\frac{1}{2}}(\varphi''')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2.
\end{aligned}$$

Also, we can derive

$$\tau|\langle\overline{g},\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0|\le\tau\|(\varphi'')^{\frac{1}{2}}\overline{g}\|_0^2+\tau\|(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2.$$

Hence, we obtain

$$
\begin{aligned}
&\tau|\langle \theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi''\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0| \\
&\leq C\Big[\tau\||\tilde{V}|^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau\||\tilde{W}|^{\frac{1}{2}}(\varphi''')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 \\
&\qquad + \tau\||\partial_t\tilde{W}|^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 \\
&\qquad + \tau^2\|(\varphi')^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}|\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau\|(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau\|(\varphi'')^{\frac{1}{2}}\overline{g}\|_0^2\Big]. \qquad (4.12)
\end{aligned}
$$

Step 5.3: Estimate the term $\tau^{-1}|\langle \theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0|$. Once again, we write $\tilde{U} := \tilde{V} - \frac{n-2s}{2}\tilde{W}$. Straightforward integration by parts implies

$$
\begin{aligned}
&|\tau^{-1}\langle \tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0| \\
&= \tau^{-1}|\langle \tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-2}\varphi''\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0 - \langle \partial_t\tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0 \\
&\qquad - \langle \tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&\leq \tau^{-1}|\langle \tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| + \tau^{-1}|\langle \partial_t\tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&\quad + \tau^{-1}\||\tilde{U}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2
\end{aligned}
$$

since $(\varphi')^{-2}\varphi'' = (\alpha e^{\alpha t})^{-2}\alpha^2 e^{\alpha t} = e^{-\alpha t} = \varphi^{-1}$. Using integration by parts again, we have

$$
\begin{aligned}
&\tau^{-1}|\langle \tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&= \tau^{-1}\Big| -\frac{1}{2}\langle \tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, -\varphi^{-2}\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0 - \frac{1}{2}\langle \partial_t\tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \varphi^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\rangle_0\Big| \\
&\leq \tau^{-1}\||\tilde{U}|^{\frac{1}{2}}\varphi^{-1}(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau^{-1}\||\partial_t\tilde{U}|^{\frac{1}{2}}\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2.
\end{aligned}
$$

Also, we can estimate

$$
\begin{aligned}
&\tau^{-1}|\langle \partial_t\tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&= \tau^{-1}|\langle \partial_t\tilde{U}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&\leq \tau^{-1}\||\partial_t\tilde{U}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau^{-1}\||\partial_t\tilde{U}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2.
\end{aligned}
$$

Combining these inequalities gives

$$
\begin{aligned}
&|\tau^{-1}\langle \tilde{U}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0| \\
&\leq C\Big[\tau^{-1}\||\tilde{U}|^{\frac{1}{2}}\varphi^{-1}(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau^{-1}\||\partial_t\tilde{U}|^{\frac{1}{2}}\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 \\
&\qquad + \tau^{-1}\||\partial_t\tilde{U}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 + \tau^{-1}\||\tilde{U}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2\Big]. \qquad (4.13)
\end{aligned}
$$

On the other hand, using the integration by parts yields

$$\begin{aligned}
&|\tau^{-1}\langle \tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0| \\
&= \tau^{-1}\left| -\frac{1}{2}\langle \tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}, -\overbrace{(\varphi')^{-2}\varphi''}^{=\varphi^{-1}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0 - \frac{1}{2}\langle \partial_t\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0\right| \\
&\le \tau^{-1}\||\tilde{W}|^{\frac{1}{2}}\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 + \tau^{-1}\||\partial_t\tilde{W}|^{\frac{1}{2}}\partial_t\overline{v}\|_0^2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&|\tau^{-1}\langle \tau\varphi'\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0| \\
&= |\langle \tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0| \\
&= | - \langle \partial_t\tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, \theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0 - \langle \tilde{W}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}, \theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&\le \||\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 + \||\partial_t\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\tau^{-1}|\langle \overline{g}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0| \\
&= \tau^{-1}| - \langle \overline{g}, -\overbrace{(\varphi')^{-2}\varphi''}^{\varphi^{-1}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0 - \langle \partial_t\overline{g}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&\le \tau^{-1}|\langle \varphi^{-\frac{1}{2}}\overline{g}, \varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| + |\langle \tau^{-1}(\varphi')^{-\frac{1}{2}}\partial_t\overline{g}, (\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\rangle_0| \\
&\le \tau^{-1}\|\varphi^{-\frac{1}{2}}\overline{g}\|_0^2 + \tau^{-1}\|\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 \\
&\quad + \tau^{-2}\|(\varphi')^{-\frac{1}{2}}\partial_t\overline{g}\|_0^2 + \|(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2
\end{aligned}$$

Combining these three inequalities with (4.13), we finally have

$$\begin{aligned}
&\tau^{-1}|\langle \theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}, (\varphi')^{-1}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t^2\overline{v}\rangle_0| \\
&\le C\Big[\tau^{-1}\||\tilde{V}|^{\frac{1}{2}}\varphi^{-1}(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau^{-1}\||\partial_t\tilde{V}|^{\frac{1}{2}}\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 \\
&\qquad + \tau^{-1}\||\tilde{W}|^{\frac{1}{2}}\varphi^{-1}(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 + \tau^{-1}\||\partial_t\tilde{W}|^{\frac{1}{2}}\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2 \\
&\qquad + \tau^{-1}\||\partial_t\tilde{V}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 + \tau^{-1}\||\tilde{V}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 \\
&\qquad + \||\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 + \||\partial_t\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 \\
&\qquad + \|\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2 + \tau^{-1}\|(\varphi')^{-\frac{1}{2}}\overline{g}\|_0^2 + \tau^{-2}\|(\varphi')^{-\frac{1}{2}}\partial_t\overline{g}\|_0^2\Big].
\end{aligned} \tag{4.14}$$

Step 6: Conclusion. It follows from (4.11), (4.12), (4.14), and (4.10), that

$$\begin{aligned}
&\tau^3\|\varphi'(\varphi'')^{\frac{1}{2}}\overline{v}\|^2+\tau\|(\varphi'')^{\frac{1}{2}}\partial_t\overline{v}\|^2+\tau\|(\varphi')^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|^2\\
&+\tau^{-1}\|(\varphi')^{-\frac{1}{2}}\partial_t^2\overline{v}\|^2+\tau^{-1}\|(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|^2\\
&\le C\bigg[\|L_\varphi\overline{v}\|^2+\tau\||\tilde{V}|^{\frac{1}{2}}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2+\tau\||\partial_t\tilde{V}|^{\frac{1}{2}}|\varphi'|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\
&\qquad+\tau^2\|(\varphi')^{\frac{1}{2}}(\varphi'')^{\frac{1}{2}}|\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2+\tau^2\||\partial_t\tilde{W}|^{\frac{1}{2}}\varphi'\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\
&\qquad+\tau^2\|(\varphi'')^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\overline{v}\|_0^2\\
&\quad+\tau^{-1}\||\partial_t\tilde{V}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2+\tau^{-1}\||\tilde{V}|^{\frac{1}{2}}(\varphi')^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2\\
&\qquad+\||\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2+\||\partial_t\tilde{W}|^{\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2\\
&\qquad+\|\varphi^{-\frac{1}{2}}\theta_{n+1}^{\frac{2s-1}{2}}\partial_t\overline{v}\|_0^2+\tau^2\|(\varphi')^{\frac{3}{2}}\overline{g}\|_0^2+\|(\varphi')^{-\frac{1}{2}}\partial_t\overline{g}\|_0^2\bigg].
\end{aligned}\tag{4.15}$$

Finally, expressing (4.15) by the Cartesian coordinates, we prove the desired estimate. □

4.2. **Carleman estimate with a non-differentiable potential.** We now derive the Carleman estimate in the case of a non-differentiable potential. The proof is modified from that of [RW19, Theorem 5], which uses the idea in [KLW16].

Theorem 4.2. *Let $\alpha>1$, $s\in(\frac{1}{2},1)$ and $\tilde{u}\in H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$ with $\operatorname{supp}(\tilde{u})\subset B_R^+\setminus B_1^+$ for some constant $R>1$ be a solution to*

$$\begin{cases}\nabla\cdot x_{n+1}^{1-2s}\nabla\tilde{u}=f & \text{in } \mathbb{R}^{n+1}_+,\\ \tilde{u}=u & \text{on } \mathbb{R}^n\times\{0\},\\ \lim_{x_{n+1}\to0}x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}=Wx'\cdot\nabla'u+Vu+g & \text{on } \mathbb{R}^n\times\{0\},\end{cases}$$

where $f\in L^2(\mathbb{R}^{n+1}_+,x_{n+1}^{2s-1})$ is compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, $g\in L^2(\mathbb{R}^n)$ is compactly supported in $\mathbb{R}^n$, $x\cdot\nabla W$ exists, and W satisfies

$$0\le W\le\lambda|x|^{-\alpha}\quad\text{and}\quad|x\cdot\nabla W|\le\lambda|x|^{-\alpha}.\tag{4.16}$$

Define $\phi(x):=|x|^\alpha$. Then there exist constants $C,\tau_0>1$ such that

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\alpha}{2}-1}x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|^2_{L^2(\mathbb{R}^{n+1}_+)}+\tau\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\
&\le C\bigg[\|e^{\tau\phi}x_{n+1}^{\frac{2s-1}{2}}|x|f\|^2_{L^2(\mathbb{R}^{n+1}_+)}+\tau^{2-2s}\|e^{\tau\phi}(|V|+|W|)|x|^{(1-\alpha)s}u\|_{L^2(\mathbb{R}^n\times\{0\})}\\
&\qquad+\|e^{\tau\phi}|x|^{(1-\alpha)s}|x|^{\beta+\frac{1}{2}}g\|^2_{L^2(\mathbb{R}^n\times\{0\})}\bigg]
\end{aligned}\tag{4.17}$$

for all $\tau\ge\tau_0$.

Proof. **Step 1: Pass to conformal polar coordinates.** As in the proof of Theorem 4.1, we first pass to conformal polar coordinates and obtain (4.2). Let $K\ge1$ be a constant to be determined later and denote $\varphi(t):=\phi(e^t)$. We split $\overline{u}$ into $\overline{u}=u_1+u_2$ and

$\operatorname{supp}(u_1), \operatorname{supp}(u_2) \subset \mathcal{S}^n_+ \times \{t>0\}$, where u_1 is a solution to

(4.18)

$$\begin{cases} \left[\theta_{n+1}^{1-2s}\partial_t^2 + \nabla_{\mathcal{S}^n}\cdot\theta_{n+1}^{1-2s}\nabla_{\mathcal{S}^n} - \theta_{n+1}^{1-2s}\dfrac{(n-2s)^2}{4} - K^2\tau^2|\varphi'|^2\theta_{n+1}^{1-2s}\right]u_1 = \tilde{f} & \text{in} \quad \mathcal{S}^n_+\times\mathbb{R}, \\ \lim\limits_{\theta_{n+1}\to 0}\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}u_1 = \left(\tilde{V} - \dfrac{n-2s}{2}\tilde{W}\right)\overline{u} + \tilde{W}\partial_t u_1 + \tilde{g} & \text{on} \quad \partial\mathcal{S}^n_+\times\mathbb{R}. \end{cases}$$

The existence of u_1 is guaranteed by the Lax-Milgram Theorem in $H^1(\mathcal{S}^n_+\times\mathbb{R}, \theta_{n+1}^{1-2s})$. Note that u_2 satisfies

(4.19)

$$\begin{cases} \left[\theta_{n+1}^{1-2s}\partial_t^2 + \nabla_{\mathcal{S}^n}\cdot\theta_{n+1}^{1-2s}\nabla_{\mathcal{S}^n} - \theta_{n+1}^{1-2s}\dfrac{(n-2s)^2}{4}\right]u_2 = -K^2\tau^2|\varphi'|^2\theta_{n+1}^{1-2s}u_1 & \text{in} \quad \mathcal{S}^n_+\times\mathbb{R}, \\ \lim\limits_{\theta_{n+1}\to 0}\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}u_2 = \tilde{W}\partial_t u_2 & \text{on} \quad \partial\mathcal{S}^n_+\times\mathbb{R}. \end{cases}$$

In what follows, we shall use the same notations as in the proof of Theorem 4.1.

Step 2: Elliptic estimate for u_1. Test (4.18) by $\tau^2 e^{2\tau\varphi}|\varphi''|^2 u_1$, we obtain [RW19, equation (42)]:

$$\begin{aligned} &\tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\partial_t u_1\|^2 + \tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_1\|^2 \\ &\quad + \tau^2\frac{(n-2s)^2}{4}\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}u_1\|^2 + K^2\tau^4\|e^{\tau\varphi}\varphi'\varphi''\theta_{n+1}^{\frac{1-2s}{2}}u_1\|^2 \\ &= -\tau^2\langle e^{2\tau\varphi}\tilde{f}, |\varphi''|^2u_1\rangle + \tau^2\langle e^{2\tau\varphi}\lim_{\theta_{n+1}\to 0}\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}u_1, |\varphi''|^2u_1\rangle_0 \\ &\quad - 2\tau^3\langle e^{2\tau\varphi}|\varphi''|^2\varphi'\theta_{n+1}^{1-2s}\partial_t u_1, u_1\rangle - 2\tau^2\langle e^{2\tau\varphi}\varphi''\varphi'''u_1, \theta_{n+1}^{1-2s}\partial_t u_1\rangle. \end{aligned}$$

Taking $K \geq 1$ sufficiently large yields

$$\begin{aligned} &\tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\partial_t u_1\|^2 + \tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_1\|^2 \\ &\quad + \frac{1}{2}K^2\tau^4\|e^{\tau\varphi}\varphi'\varphi''\theta_{n+1}^{\frac{1-2s}{2}}u_1\|^2 \\ &\leq C\Big[\|e^{\tau\varphi}\theta_{n+1}^{\frac{2s-1}{2}}\tilde{f}\|^2 + \tau^2|\langle e^{2\tau\varphi}\lim_{\theta_{n+1}\to 0}\theta_{n+1}^{1-2s}\nu\cdot\nabla_{\mathcal{S}^n}u_1, |\varphi''|^2u_1\rangle_0|\Big]. \end{aligned} \tag{4.20}$$

Note that

$$\begin{aligned} &\tau^2\langle e^{2\tau\varphi}\tilde{W}\partial_t u_1, |\varphi''|^2 u_1\rangle_0 \\ &= -2\tau^3\langle\varphi' e^{2\tau\varphi}\tilde{W}u_1, |\varphi''|^2u_1\rangle_0 - \tau^2\langle e^{2\tau\varphi}(\partial_t\tilde{W})u_1, |\varphi''|^2u_1\rangle_0 \\ &\quad - 2\tau^2\langle e^{2\tau\varphi}\tilde{W}u_1, \varphi''\varphi'''u_1\rangle_0 - \overbrace{\tau^2\langle e^{2\tau\varphi}\tilde{W}u_1, |\varphi''|^2\partial_t u_1\rangle_0}^{\text{identical to LHS}}, \end{aligned}$$

which gives

$$
\begin{aligned}
&\tau^2\langle e^{2\tau\varphi}\tilde{W}\partial_t u_1, |\varphi''|^2 u_1\rangle_0 \\
&= -\tau^3\langle \varphi' e^{2\tau\varphi}\tilde{W}u_1, |\varphi''|^2 u_1\rangle_0 - \frac{1}{2}\tau^2\langle e^{2\tau\varphi}(\partial_t\tilde{W})u_1, |\varphi''|^2 u_1\rangle_0 \\
&\quad - \tau^2\langle e^{2\tau\varphi}\tilde{W}u_1, \varphi''\varphi''' u_1\rangle_0 \\
&= -\tau^3\|e^{\tau\varphi}|\varphi'|^{\frac12}\varphi''|\tilde{W}|^{\frac12}u_1\|_0^2 - \frac{1}{2}\tau^2\|e^{\tau\varphi}|\partial_t\tilde{W}|^{\frac12}u_1\|_0^2 \\
&\quad - \tau^2\|e^{\tau\varphi}|\varphi''|^{\frac12}|\varphi'''|^{\frac12}|\tilde{W}|^{\frac12}u_1\|_0^2.
\end{aligned}
$$

Since $s > \frac{1}{2}$, given any $\epsilon > 0$, we notice that $\tau^3 \le \epsilon\tau^{2+2s}$ for all sufficiently large $\tau > 1$ and, hence,

$$
\begin{aligned}
&|\tau^2\langle e^{2\tau\varphi}\tilde{W}\partial_t u_1, |\varphi''|^2 u_1\rangle_0| \\
&\le \epsilon\tau^{2+2s}\left[\|e^{\tau\varphi}|\varphi'|^{\frac12}\varphi''|\tilde{W}|^{\frac12}u_1\|_0^2 + \|e^{\tau\varphi}|\partial_t\tilde{W}|^{\frac12}u_1\|_0^2\right].
\end{aligned}
$$

In view of the inequality above, (4.20) becomes

$$
\begin{aligned}
&\tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\partial_t u_1\|^2 + \tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_1\|^2 + \frac{1}{2}K^2\tau^4\|e^{\tau\varphi}\varphi'\varphi''\theta_{n+1}^{\frac{1-2s}{2}}u_1\|^2 \\
&\le C\left[\|e^{\tau\varphi}\theta_{n+1}^{\frac{2s-1}{2}}\tilde{f}\|^2 + \tau^{2-2s}\|e^{\tau\varphi}\varphi''(|\tilde{V}|+|\tilde{W}|)e^{-\alpha st}\overline{u}\|_0^2 + \tau^{2-2s}\|e^{\tau\varphi}\varphi''e^{-\alpha st}\tilde{g}\|_0^2\right] \\
&\quad + \epsilon\tau^{2+2s}\left[\|e^{\tau\varphi}\varphi''e^{\alpha st}u_1\|_0^2 + \|e^{\tau\varphi}|\varphi'|^{\frac12}\varphi''|\tilde{W}|^{\frac12}u_1\|_0^2 + \|e^{\tau\varphi}|\partial_t\tilde{W}|^{\frac12}u_1\|_0^2\right].
\end{aligned} \tag{4.21}
$$

From (4.16), we have $|\varphi'|^{\frac12}|\tilde{W}|^{\frac12} \le \sqrt{\lambda\alpha}$ and $|\partial_t\tilde{W}|^{\frac12} \le \sqrt{\lambda}\varphi''$ in supp (u_1). By the trace inequality in Remark A.4 (see also [RW19, (41)]) and choosing ϵ sufficiently small, the last three terms in (4.21) can be absorbed into its left-hand-side, that is,

$$
\begin{aligned}
&\tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\partial_t u_1\|^2 + \tau^2\|e^{\tau\varphi}\varphi''\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_1\|^2 + \frac{1}{2}K^2\tau^4\|e^{\tau\varphi}\varphi'\varphi''\theta_{n+1}^{\frac{1-2s}{2}}u_1\|^2 \\
&\le C\left[\|e^{\tau\varphi}\theta_{n+1}^{\frac{2s-1}{2}}\tilde{f}\|^2 + \tau^{2-2s}\|e^{\tau\varphi}\varphi''(|\tilde{V}|+|\tilde{W}|)e^{-\alpha st}\overline{u}\|_0^2 + \tau^{2-2s}\|e^{\tau\varphi}\varphi''e^{-\alpha st}\tilde{g}\|_0^2\right],
\end{aligned} \tag{4.22}
$$

which holds for all sufficiently large $\tau > 1$.

Step 3: Subelliptic estimate for u_2. Applying Theorem 4.1 with $V = 0$ and $g = 0$, together with (4.16), we have

$$
\begin{aligned}
&\tau^3\|e^{\tau\varphi}\varphi'|\varphi''|^{\frac12}\theta_{n+1}^{\frac{1-2s}{2}}u_2\|^2 \\
&\quad + \tau\|e^{\tau\varphi}|\varphi''|^{\frac12}\theta_{n+1}^{\frac{1-2s}{2}}\partial_t u_2\|^2 + \tau\|e^{\tau\varphi}|\varphi''|^{\frac12}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_2\| \\
&\quad + \tau^{-1}\|e^{\tau\varphi}(\varphi')^{-1}|\varphi''|^{\frac12}\theta_{n+1}^{\frac{1-2s}{2}}\partial_t^2 u_2\|^2 + \tau^{-1}\|e^{\tau\varphi}(\varphi')^{-1}|\varphi''|^{\frac12}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\partial_t u_2\|^2 \\
&\le C\left[K^4\tau^4\|e^{\tau\varphi}|\varphi'|^2\theta_{n+1}^{\frac{1-2s}{2}}u_1\|^2 + \tau^2\|e^{\tau\varphi}|\varphi''|^{\frac12}u_2\|_0^2 + \|e^{\tau\varphi}\varphi^{-\frac12}\partial_t u_2\|_0\right].
\end{aligned} \tag{4.23}
$$

Again, by the trace inequality in Remark A.4, we can see that

$$
\tau^2\|e^{\tau\varphi}|\varphi''|^{\frac12}u_2\|_0^2 \le C\left[\tau^{4-2s}\|e^{\tau\varphi}|\varphi''|^{\frac12}\theta_{n+1}^{\frac{1-2s}{2}}u_2\|^2 + \tau^{2-2s}\|e^{\tau\varphi}|\varphi''|^{\frac12}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_2\|^2\right].
$$

Since $s > \frac{1}{2}$, $\tau^2 \|e^{\tau\varphi}|\varphi''|^{\frac{1}{2}} u_1\|_0^2$ can be absorbed by the left-hand-side of (4.23). Likewise, the term $\|e^{\tau\varphi}\varphi^{-\frac{1}{2}}\partial_t u_2\|_0$ can be dropped using a similar trace inequality. So, for large $\tau > 1$, we reach

$$\tau^3\|e^{\tau\varphi}\varphi'|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}u_2\|^2 + \tau\|e^{\tau\varphi}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\partial_t u_2\|^2 + \tau\|e^{\tau\varphi}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_2\|^2$$
$$\leq CK^4\tau^4\|e^{\tau\varphi}|\varphi'|^2\theta_{n+1}^{\frac{1-2s}{2}}u_1\|^2. \tag{4.24}$$

Step 4: Conclusion. Combining (4.22) and (4.24) leads to

$$\tau^3\|e^{\tau\varphi}\varphi'|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\overline{u}\|^2 + \tau\|e^{\tau\varphi}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\partial_t\overline{u}\|^2 + \tau\|e^{\tau\varphi}|\varphi''|^{\frac{1}{2}}\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\overline{u}\|^2$$
$$\leq CK^4\left[\|e^{\tau\varphi}\theta_{n+1}^{\frac{2s-1}{2}}\tilde{f}\|^2 + \tau^{2-2s}\|e^{\tau\varphi}\varphi''(|\tilde{V}| + |\tilde{W}|)e^{-\alpha st}\overline{u}\|_0^2 + \tau^{2-2s}\|e^{\tau\varphi}\varphi''e^{-\beta st}\tilde{g}\|_0^2\right]. \tag{4.25}$$

and the required estimate follows by expressing the estimate above in the original coordinates. □

5. Proof of Landis-type Conjecture

We prove the main theorems in this section. We will discuss the cases of differentiable and non-differentiable potentials separately.

5.1. Differentiable potentials.

Proof of Theorem 1.1. **Step 1: Apply Carleman estimate.** Let $\eta_R \in \mathcal{C}^\infty(\overline{\mathbb{R}^{n+1}_+})$ be a radial cut-off function satisfying that there exists a constant $C > 1$, which is independent of $R > 2$, such that

$$\eta_R(x) = \begin{cases} 0, & |x| \leq 1 \text{ or } |x| \geq 2R, \\ 1, & 2 \leq |x| \leq R, \end{cases}$$

with

$$\begin{cases} |\nabla\eta_R| \leq C, \quad |\nabla^2\eta_R| \leq C, & \text{in } A^+_{1,2}, \\ |\nabla\eta_R| \leq C/R, \quad |\nabla^2\eta_R| \leq C/R^2 & \text{in } A^+_{R,2R}. \end{cases} \tag{5.1}$$

Define $w := \eta_R\tilde{u}$. By using radial dependence of η_R, we have

$$\nabla\cdot x_{n+1}^{1-2s}\nabla w = f \quad \text{in } \mathbb{R}^{n+1}_+,$$

where

$$f = 2x_{n+1}^{1-2s}\nabla\eta_R\cdot\nabla\tilde{u} + \tilde{u}\nabla\cdot x_{n+1}^{1-2s}\nabla\eta_R \in L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{2s-1}), \tag{5.2}$$

and

$$\begin{aligned}
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}w &= \eta_R \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u} + \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}(\partial_{n+1}\eta_R)\tilde{u} \\
&= \eta_R\left[bx'\cdot\nabla' u + qu\right] + \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}(\partial_r\eta_R\partial_{n+1}(|x|))\tilde{u} \\
&= bx'\cdot\nabla' w + qw + g + \lim_{x_{n+1}\to 0} x_{n+1}^{2-2s}(\partial_r\eta_R)\frac{1}{|x|}\tilde{u} \\
&= bx'\cdot\nabla' w + qw + g
\end{aligned}$$

with $g = -(x'\cdot\nabla'\eta_R)bu$. Choose α with $1<\alpha<\beta$. Plugging these functions into the Carleman inequality (4.1) with $\phi(x)=|x|^\alpha$, together with (1.3), and (1.4), we have

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|_{L^2(\mathbb{R}^{n+1}_+)}^2+\tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}^2\\
&\quad+\tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla(x\cdot\nabla w)\|_{L^2(\mathbb{R}^{n+1}_+)}^2\\
&\le C\Big[\|e^{\tau\phi}|x|x_{n+1}^{\frac{2s-1}{2}}f\|_{L^2(\mathbb{R}^{n+1}_+)}^2\\
&\quad+\tau^2\|e^{\tau\phi}|x|^{\frac{\beta}{2}}w\|_{L^2(\mathbb{R}^n\times\{0\})}^2+\|e^{\tau\phi}|x|^{-\frac{\beta}{2}}(x'\cdot\nabla'w)\|_{L^2(\mathbb{R}^n\times\{0\})}^2\\
&\quad+\tau^2\|e^{\tau\phi}|x|^{\frac{3\beta}{2}}|x|^{\frac{1+2s}{2}}(x'\cdot\nabla'\eta_R)bu\|_{L^2(\mathbb{R}^n\times\{0\})}^2\\
&\quad+\|e^{\tau\phi}|x|^{-\frac{3\beta}{2}}|x|^s x'\cdot\nabla'((x'\cdot\nabla'\eta_R)bu)\|_{L^2(\mathbb{R}^n\times\{0\})}^2\Big].
\end{aligned}\tag{5.3}$$

Step 2: Estimate the bulk contribution. Note that $1\le\frac{|x|}{R}$ in $A^+_{R,2R}$ and $1\le|x|$ in $A^+_{1,2}$. Then we have

$$\begin{aligned}
&\|e^{\tau\phi}|x|x_{n+1}^{\frac{2s-1}{2}}f\|_{L^2(\mathbb{R}^{n+1}_+)}^2\\
&\le C\Big[R^{-4}\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\tilde u\|_{L^2(A^+_{R,2R})}^2+R^{-2}\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\nabla\tilde u\|_{L^2(A^+_{R,2R})}^2\\
&\quad+\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\tilde u\|_{L^2(A^+_{1,2})}^2+\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\nabla\tilde u\|_{L^2(A^+_{1,2})}^2\Big].
\end{aligned}$$

Combining Proposition 3.1 and the arguments in [Kow19, RW19], we can show

$$\lim_{R\to+\infty}\Big[R^{-4}\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\tilde u\|_{L^2(A^+_{R,2R})}^2+R^{-2}\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\nabla\tilde u\|_{L^2(A^+_{R,2R})}^2\Big]=0.$$

It follows from (5.1) and (1.5) that

$$\begin{aligned}
&\limsup_{R\to+\infty}\Big[\tau^2\|e^{\tau\phi}|x|^{\frac{3\beta}{2}}|x|^{\frac{1+2s}{2}}(x'\cdot\nabla'\eta_R)bu\|_{L^2(\mathbb{R}^n\times\{0\})}^2\\
&\qquad+\|e^{\tau\phi}|x|^{-\frac{3\beta}{2}}|x|^s x'\cdot\nabla'((x'\cdot\nabla'\eta_R)bu)\|_{L^2(\mathbb{R}^n\times\{0\})}^2\Big]\\
&\le C\Big[\tau^2\|e^{\tau\phi}u\|_{L^2(A'_{1,2})}^2+\|e^{\tau\phi}\nabla'u\|_{L^2(A'_{1,2})}^2\Big].
\end{aligned}$$

Taking $R\to+\infty$ in (5.3) implies

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|_{L^2(\mathbb{R}^{n+1}_+)}^2+\tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}^2\\
&\quad+\tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla(x\cdot\nabla w)\|_{L^2(\mathbb{R}^{n+1}_+)}^2\\
&\le C\Big[\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\tilde u\|_{L^2(A^+_{1,2})}^2+\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\nabla\tilde u\|_{L^2(A^+_{1,2})}^2+\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}^2\\
&\quad+\tau^2\|e^{\tau\phi}|x|^{\frac{\beta}{2}}w\|_{L^2(\mathbb{R}^n\times\{0\})}^2+\|e^{\tau\phi}|x|^{-\frac{\beta}{2}}(x'\cdot\nabla'w)\|_{L^2(\mathbb{R}^n\times\{0\})}^2\\
&\quad+\tau^2\|e^{\tau\phi}u\|_{L^2(A'_{1,2})}^2+\|e^{\tau\phi}\nabla'u\|_{L^2(A'_{1,2})}^2\Big].
\end{aligned}\tag{5.4}$$

Step 3: Estimate the boundary contribution. By the trace inequality in Remark A.4 (also see the similar arguments in [Kow19, RW19]), we can derive following two inequalities,

$$\begin{aligned}&\tau^{2s+1}\|e^{\tau\phi}|x|^{\frac{\beta}{2}}w\|^2_{L^2(\mathbb{R}^n\times\{0\})}\\&\leq C\Big[\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|^2_{L^2(\mathbb{R}^{n+1}_+)}+\tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)}\Big]\end{aligned}$$

and

$$\begin{aligned}&\tau^{2s-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}}(x'\cdot\nabla' w)\|^2_{L^2(\mathbb{R}^n\times\{0\})}\\&\leq C\Big[\tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)}+\tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla(x\cdot\nabla w)\|^2_{L^2(\mathbb{R}^{n+1}_+)}\Big].\end{aligned}$$

By these two inequalities, for large τ, the boundary terms of (5.4) can be ignored and we obtain

$$\begin{aligned}&\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|^2_{L^2(B_6^+\setminus B_4^+)}\\&\leq\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|^2_{L^2(\mathbb{R}^{n+1}_+)}+\tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\&\quad+\tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla(x\cdot\nabla w)\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\&\leq C\Big[\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\tilde u\|^2_{L^2(A^+_{1,2})}+\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\nabla\tilde u\|^2_{L^2(A^+_{1,2})}\\&\quad+\tau^2\|e^{\tau\phi}u\|^2_{L^2(A'_{1,2})}+\|e^{\tau\phi}\nabla' u\|^2_{L^2(A'_{1,2})}\Big].\end{aligned}\tag{5.5}$$

Pulling out the exponential weight in (5.5) and using $w=\tilde u$ in $B_6^+\setminus B_4^+$, we have

$$\begin{aligned}&\tau^3e^{\tau\phi(4)}\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|^2_{L^2(B_6^+\setminus B_4^+)}\\&\leq Ce^{\tau\phi(2)}\Big[\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|^2_{L^2(A^+_{1,2})}+\|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde u\|^2_{L^2(A^+_{1,2})}+\tau^2\|u\|^2_{L^2(A'_{1,2})}+\|\nabla' u\|^2_{L^2(A'_{1,2})}\Big].\end{aligned}$$

It is obvious that $e^{\tau\phi(4)}\geq e^{\tau\phi(2)}$ and thus

$$\begin{aligned}&\tau\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|^2_{L^2(B_6^+\setminus B_4^+)}\\&\leq C\Big[\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|^2_{L^2(A^+_{1,2})}+\|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde u\|^2_{L^2(A^+_{1,2})}+\|u\|^2_{L^2(A'_{1,2})}+\|\nabla' u\|^2_{L^2(A'_{1,2})}\Big].\end{aligned}$$

If $\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|^2_{L^2(B_6^+\setminus B_4^+)}\neq 0$, then

$$\lim_{\tau\to+\infty}\tau\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|^2_{L^2(B_6^+\setminus B_4^+)}=+\infty,$$

which is a contradiction. Thus, we must have $\tilde u=0$ in $B_6^+\setminus B_4^+$, i.e., $u=0$ in $B_6'\setminus B_4'$. Finally, by the unique continuation property [GSU20], we conclude that $u\equiv 0$. ☐

5.2. **Non-differentiable potentials.**

Proof of Theorem 1.2. Let η_R be the cut-off function as in the proof of Theorem 1.1. Choose $\beta>\alpha>\frac{4s}{4s-1}$. Let $\overline{w}(t,\theta)=\overline{u}(t,\theta)\eta_R(e^t\theta)$, where $\overline{u}:=e^{\frac{n-2s}{2}t}\tilde u$. In the proof, it will be more convenient to apply (4.25) with $\varphi(t) = e^{\alpha t}$ rather than (4.17). Hence, we can get

$$\begin{aligned}
&\tau^3 \| e^{\tau\varphi} \varphi' |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \overline{w} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} \\
&\quad + \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \partial_t \overline{w} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} + \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} \overline{w} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} \\
&\le C \Big[\| e^{\tau\varphi} \theta_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} + \tau^{2-2s} \| e^{\tau\varphi} \varphi'' (|\tilde{q}| + |\tilde{b}|) e^{-\alpha s t} \overline{w} \|^2_{L^2(\partial \mathcal{S}^n_+ \times \mathbb{R})} \\
&\qquad + \tau^{2-2s} \| e^{\tau\varphi} \varphi'' e^{-\alpha s t} \tilde{g} \|^2_{L^2(\partial \mathcal{S}^n_+ \times \mathbb{R})} \Big],
\end{aligned} \tag{5.6}$$

where $\tilde{q} = e^{2st} q$, $\tilde{b} = e^{2st} b$, $\tilde{g} = -(\partial_t \eta_R) \tilde{b} \overline{w}$, and

$$\tilde{f} = e^{\frac{n+2+2s}{2} t} f,$$

where f is given in (5.2). Recall in the proof of Theorem 1.1, we have shown that

$$\begin{aligned}
&\limsup_{R\to\infty} \| e^{\tau\varphi} \theta_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} \\
&\le C \Big[\| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \partial_t \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} + \| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} + \| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} \Big]
\end{aligned}$$

and

$$\begin{aligned}
&\tau^{2-2s} \| e^{\tau\varphi} \varphi'' e^{-\alpha s t} \tilde{g} \|^2_{L^2(\partial \mathcal{S}^n_+ \times \mathbb{R})} \\
&\le C \Big[\tau^{4-2s} \| e^{\tau\phi} u \|^2_{L^2(A'_{1,2})} + \tau^{2-2s} \| e^{\tau\phi} \nabla' u \|^2_{L^2(A'_{1,2})} \Big].
\end{aligned}$$

Following the exactly same argument as in the proof of [RW19, Theorem 2] and taking into account the boundedness of q and b in the support of $\overline{w}$, the boundary term

$$\tau^{2-2s} \| e^{\tau\varphi} \varphi'' (|\tilde{q}| + |\tilde{b}|) e^{-\alpha s t} \overline{w} \|^2_{L^2(\partial \mathcal{S}^n_+ \times \mathbb{R})}$$

can be absorbed by the left-hand-side of (5.6), that is, we obtain

$$\begin{aligned}
&\tau^3 \| e^{\tau\varphi} \varphi' |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \overline{w} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} \\
&\quad + \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \partial_t \overline{w} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} + \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} \overline{w} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} \\
&\le C \Big[\| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \partial_t \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} + \| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} \\
&\qquad + \| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} + \tau^{4-2s} \| e^{\tau\phi} u \|^2_{L^2(A'_{1,2})} + \tau^{2-2s} \| e^{\tau\phi} \nabla' u \|^2_{L^2(A'_{1,2})} \Big].
\end{aligned} \tag{5.7}$$

Next, from (5.7), it follows that

$$\begin{aligned}
&\tau^3 \| e^{\tau\varphi} \varphi' |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [4,6])} \\
&\le \tau^3 \| e^{\tau\varphi} \varphi' |\varphi''|^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \overline{w} \|^2_{L^2(\mathcal{S}^n_+ \times \mathbb{R})} \\
&\le C \tau^{4-2s} \Big[\| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \partial_t \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} + \| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} \\
&\qquad + \| e^{\tau\varphi} \theta_{n+1}^{\frac{1-2s}{2}} \overline{u} \|^2_{L^2(\mathcal{S}^n_+ \times [1,2])} + \| e^{\tau\phi} u \|^2_{L^2(A'_{1,2})} + \| e^{\tau\phi} \nabla' u \|^2_{L^2(A'_{1,2})} \Big].
\end{aligned}$$

Pulling out the weight $e^{\tau\phi}$ gives

$$\begin{aligned} &e^{\tau\varphi(4)}\tau^3\|\theta_{n+1}^{\frac{1-2s}{2}}\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[4,6])} \\ &\leq Ce^{\tau\varphi(2)}\tau^{4-2s}\Big[\|\theta_{n+1}^{\frac{1-2s}{2}}\partial_t\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[1,2])}+\|\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[1,2])} \\ &\qquad +\|\theta_{n+1}^{\frac{1-2s}{2}}\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[1,2])}+\|u\|^2_{L^2(A'_{1,2})}+\|\nabla' u\|^2_{L^2(A'_{1,2})}\Big]. \end{aligned}$$

Clearly, $e^{\tau\varphi(4)}\leq e^{\tau\varphi(2)}$ and thus

$$\begin{aligned} &\tau^{2s-1}\|\theta_{n+1}^{\frac{1-2s}{2}}\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[4,6])} \\ &\leq C\Big[\|\theta_{n+1}^{\frac{1-2s}{2}}\partial_t\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[1,2])}+\|\theta_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[1,2])} \\ &\qquad +\|\theta_{n+1}^{\frac{1-2s}{2}}\overline{u}\|^2_{L^2(\mathcal{S}^n_+\times[1,2])}+\|u\|^2_{L^2(A'_{1,2})}+\|\nabla' u\|^2_{L^2(A'_{1,2})}\Big] \\ &<\infty. \end{aligned}$$

Since $s>\frac{1}{2}$, letting $\tau\to\infty$, we conclude $\theta_{n+1}^{\frac{1-2s}{2}}\overline{u}=0$ in $\mathcal{S}^n_+\times[4,6]$ and so is $\overline{u}$. Finally, the unique continuation property yields $u\equiv 0$. □

Appendix A. Auxiliary Lemmas

The following Caccioppoli's inequality can be found in [RW19, Lemma 2.1].

Lemma A.1. *Let $s\in(0,1)$ and $\tilde{u}\in H^1(B^+_{4r},x_{n+1}^{1-2s})$ be a solution to (2.1). Then*

$$\|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(B^+_r)}\leq C\Big[r^{-1}\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B^+_{2r})}+\|u\|^{1/2}_{L^2(B'_{2r})}\Big\|\lim_{x_{n+1}\to 0}x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\Big\|^{1/2}_{L(B'_{2r})}\Big]$$

for some constant $C=C(n,s)>0$.

By the technique of Moser iteration, De Giorgi-Nash-Moser type theorems were established in [JLX14, Proposition 2.6(a)]. Here we list the specific case that is useful in the paper.

Lemma A.2. *Suppose $a_1,a_2\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p>\frac{n}{2s}$. Let $\tilde{u}\in H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$ be a weak solution to*

$$\begin{cases} \nabla\cdot x_{n+1}^{1-2s}\nabla\tilde{u}=0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u}=u & \text{on } \mathbb{R}^n\times\{0\}, \\ c_{n,s}\lim\limits_{x_{n+1}\to 0}x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}=a_1(x')u+a_2(x') & \text{on } \mathbb{R}^n\times\{0\}. \end{cases}$$

Then there exists a constant $C=C(n,s,p,\|a_1\|_{L^p(B'_1)})>0$ such that

$$\|\tilde{u}\|_{L^\infty(B^+_{1/4})}\leq C\Big[\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B^+_1)}+\|a_2\|_{L^p(B'_1)}\Big].$$

The following interpolation inequality was proved in [RW19, Proposition 2.5], which plays an important role in our proof.

Lemma A.3. *Let $\sigma\in(0,1)$ and $v:\mathcal{S}^n_+\to\mathbb{R}$ with $v\in H^1(\mathcal{S}^n_+,\theta_{n+1}^{1-2\sigma})$. Then there exists a constant $C=C(n,\sigma)>0$ such that*

$$\|v\|_{L^2(\partial\mathcal{S}^n_+)}\leq C\Big[\beta^{1-\sigma}\|\theta_{n+1}^{\frac{1-2s}{2}}v\|_{L^2(\mathcal{S}^n_+)}+\beta^{-\sigma}\|\theta_{n+1}^{\frac{1-2\sigma}{2}}\nabla_{\mathcal{S}^n}v\|_{L^2(\mathcal{S}^n_+)}\Big]$$

for all $\beta > 1$.

Remark A.4. *Applying Lemma A.3 to* $v(\bullet) = u(t, \bullet)$ *for* $t \in \mathbb{R}$, *we have*

$$\int_{\partial\mathcal{S}^n_+} |u(t,\theta)|^2 \, d\theta \leq C\left[\beta^{2-2\sigma} \int_{\mathcal{S}^n_+} \theta_{n+1}^{1-2s} |u(t,\theta)|^2 \, d\theta + \beta^{-2\sigma} \int_{\mathcal{S}^n_+} \theta_{n+1}^{1-2s} |\nabla_{\mathcal{S}^n} u(t,\theta)|^2 \, d\theta\right].$$

Multiplying the inequality above by a weight function $w(t) \geq 0$ *and then integrating the resulting inequality in* t *yields*

$$\|w^{\frac{1}{2}} u\|_0^2 \leq C\left[\beta^{2-2\sigma} \|w^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} u\|^2 + \beta^{-2\sigma} \|w^{\frac{1}{2}} \theta_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} u\|^2\right].$$

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