Complex analysis
Lecture notes, Fall 2023
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Pu-Zhao Kow

Department of Mathematical Sciences, National Chengchi University
Email address: pzkow@g.nccu.edu.tw

Preface

The main theme of this lecture note is to explain

- the equivalence of analyticity (complex differentiability), Cauchy-Riemann equation, and the power series representation;
- Morera theorems (Section 4.7.1); and
- the Cauchy residual theorem (Theorem 5.3.6).

This lecture note is prepared for the course *Complex Analysis* during Fall Semester 2023 (112-1), which gives an introduction to complex numbers and functions, mainly based on [**BN10**], but not following the order. The e-book is available in `https://www.lib.nccu.edu.tw` (NCCU library). The lecture note may updated during the course.

Title. Complex Analysis (Fall Semester 2023, 3 credits)

Lectures. Tuesday (14:10–15:00) and Thursday (14:10–15:00, 15:10–16:00). Begins at September 12, 2023 and ends at January 12, 2024.

Language. Chinese and English. Materials will be prepared in English.

Instructor. Pu-Zhao Kow (Email: `pzkow@g.nccu.edu.tw`)

Office hour. Thursday (16:10–17:00)

Teaching Assistant. Yu-Wei Wang (Email: `111751010@nccu.edu.tw`)

TA class. Tuesday (15:10–16:00)

Completion. Homework Assignments 60%, Midterm Exam 20% (hope that at least Chapter 1–Chapter 4 are covered), Final Exam 20%

Acknowledgments. I would like to give special thanks to students who pointed out my mistakes in this note.

Exam rules.

(1) Textbook or other materials are not allowed to be used during exams.
(2) All electronic devices (including calculator, smartphone, pad, computer, ...) are prohibited during exams.
(3) One also not allowed to bring your own extra paper. TA will provide answer sheets.
(4) Before go to washroom, one must inform us before do so.
(5) If you violate one of the above rule, we will immednate terminate your writing and the marks of the exam/quiz will be 0.
(6) One must show student card or national identity card or national health insurance card or passport or resident certificate (driving license not accepted) for verification. Before the exam begins, TA should reminds all of you to bring it. If one fails to show it during exam, we consider this as a cheating and the marks of the quiz will be 0.

Contents

CHAPTER 1

The complex numbers

1.1. Definition of complex plane $\mathbb{C}$

We shall introduce the complex plane using a rather simple (and direct) way. Given a number $x \in \mathbb{R}_{\geq 0}$, it is well-known that the square root $\sqrt{x}$ of x is well-defined, which satisfies

$$(\sqrt{x})^2 = \sqrt{x} \cdot \sqrt{x} = x \quad \text{for all } x \geq 0. \tag{1.1.1}$$

This arises a natural question: It is possible to extend (1.1.1) for all $x \in \mathbb{R}$? Or, we shall ask: How to define $\mathbf{i} \equiv \sqrt{-1}$? Clearly, we should expect that

$$\mathbf{i}^2 = -1. \tag{1.1.2}$$

We will answer this question in Section 4.4.

Formally, we expect the linearity

$$(a + \mathbf{i}b) + (c + \mathbf{i}d) = (a + c) + \mathbf{i}(b + d) \quad \text{for all } a, b, c, d \in \mathbb{R}. \tag{1.1.3a}$$

By using the formal identity (1.1.2), we also formally computed that

$$\begin{aligned} (a + \mathbf{i}b) \cdot (c + \mathbf{i}d) &= ac + \mathbf{i}bc + \mathbf{i}ad + \mathbf{i}^2 bd \\ &= (ac - bd) + \mathbf{i}(ad + bc) \quad \text{for all } a, b, c, d \in \mathbb{R}. \end{aligned} \tag{1.1.3b}$$

At this point, we not yet define the element $\mathbf{i}$, therefore the identities (1.1.3a)–(1.1.3b) are not yet well-defined. However, we can rephrase (1.1.3a)–(1.1.3b) without involving the formal element $\mathbf{i}$ (which is not yet well-defined).

Definition 1.1.1. We define the set $\mathbb{C} := \mathbb{R} \times \mathbb{R} \equiv \{(x, y) : x, y \in \mathbb{R}\}$. We define the binomial operations "$+$" and "$\cdot$" on $\mathbb{C}$ by

$$\mathbb{C} + \mathbb{C} \to \mathbb{C}, \quad (a, b) + (c, d) := (a + c, b + d), \tag{1.1.4a}$$

$$\mathbb{C} \cdot \mathbb{C} \to \mathbb{C}, \quad (a, b) \cdot (c, d) := (ac - bd + ad + bc). \tag{1.1.4b}$$

Proposition 1.1.2. *$(\mathbb{C}, +, \cdot)$ is a field with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$.*

Remark 1.1.3. The main point here is to define what is the meaning of "divide an element by another element". Here the multiplication is sometimes called the *complex multiplication*, not the inner product of $\mathbb{R}^n$. While reading research articles, remember to make sure the definition of the multiplication (for example, the $\cdot$ in the CGO solution means inner product [**Sal08**]).

Proof of Proposition 1.1.2. Verify $(\mathbb{C}, +)$ forms a commutative group with additivity identity $(0, 0)$.

Additive associativity. $((a_1, b_1) + (a_2, b_2)) + (a_3, b_3) = (a_1, b_1) + ((a_2, b_2) + (a_3, b_3))$

Additive identity. $(a, b) + (0, 0) = (0, 0) + (a, b) = (a, b)$

Additive inverse element. One can easily verify that the additive inverse of (a, b) is $(-a, -b)$:

$$(a, b) + (-a, -b) = (-a, -b) + (a, b) = (0, 0).$$

In other words, $-(a, b) = (-a, -b)$.
Additive commutative. $(a, b) + (c, d) = (c, d) + (a, b)$

We now verify some properties of the multiplication operator.
Multiplicative associativity. One can directly verify that

$$\begin{aligned}
&((a_1, b_1) \cdot (a_2, b_2)) \cdot (a_3, b_3) \\
&\quad = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) \cdot (a_3, b_3) \\
&\quad = (\overset{\boxed{1}}{\boxed{a_1a_2a_3}} - \overset{\boxed{2}}{\boxed{a_3b_1b_2}} - \overset{\boxed{3}}{\boxed{a_1b_2b_3}} - \overset{\boxed{4}}{\boxed{a_2b_1b_3}}, \underset{\boxed{5}}{\boxed{a_1a_2b_3}} - \underset{\boxed{6}}{\boxed{b_1b_2b_3}} + \underset{\boxed{7}}{\boxed{a_1a_3b_2}} + \underset{\boxed{8}}{\boxed{a_2a_3b_1}}),
\end{aligned}$$

and

$$\begin{aligned}
&(a_1, b_1) \cdot ((a_2, b_2) \cdot (a_3, b_3)) \\
&\quad = (a_1, b_1) \cdot (a_2a_3 - b_2b_3, a_2b_3 + a_3b_2) \\
&\quad = (\overset{\boxed{1}}{\boxed{a_1a_2a_3}} - \overset{\boxed{3}}{\boxed{a_1b_2b_3}} - \overset{\boxed{4}}{\boxed{a_2b_1b_3}} - \overset{\boxed{2}}{\boxed{a_3b_1b_2}}, \underset{\boxed{5}}{\boxed{a_1a_2b_3}} + \underset{\boxed{7}}{\boxed{a_1a_3b_2}} + \underset{\boxed{8}}{\boxed{a_2a_3b_1}} - \underset{\boxed{6}}{\boxed{b_1b_2b_3}}),
\end{aligned}$$

therefore $((a_1, b_1) \cdot (a_2, b_2)) \cdot (a_3, b_3) = (a_1, b_1) \cdot ((a_2, b_2) \cdot (a_3, b_3))$.
Multiplicative identity. $(a, b) \cdot (1, 0) = (1, 0) \cdot (a, b) = (a, b)$
Multiplicative inverse element. For each $(a, b) \neq (0, 0)$, we define

$$(a, b)^{-1} := \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right). \tag{1.1.5}$$

We see that

$$(a, b) \cdot (a, b)^{-1} = \left(a \cdot \frac{a}{a^2 + b^2} - b \cdot \frac{-b}{a^2 + b^2}, b \cdot \frac{a}{a^2 + b^2} + a \cdot \frac{-b}{a^2 + b^2} \right) = (1, 0)$$

as well as $(a, b)^{-1} \cdot (a, b) = (1, 0)$.
Multiplicative commutative. $(a, b) \cdot (c, d) = (c, d) \cdot (a, b)$

The above four axioms imply that $(\mathbb{C} \setminus \{(0, 0)\}, \cdot)$ forms a commutative group. We have one more axiom to verify:
Distributive laws. This properties describe how the additive operator and multiplicative operator act together. We compute that

$$\begin{aligned}
&(a_1, b_1) \cdot ((a_2, b_2) + (a_3, b_3)) \\
&\quad = (a_1, b_1) \cdot (a_2 + a_3 . b_2 + b_3) \\
&\quad = (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3, a_1b_2 + a_1b_3 + b_1a_2 + b_1a_3) \\
&\quad = (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) + (a_1a_3 - b_1b_3, a_1b_3 + b_1a_3) \\
&\quad = (a_1, b_1) \cdot (a_2, b_2) + (a_1, b_1) \cdot (a_3, b_3)
\end{aligned}$$

and the multiplicative commutative also gives us that

$$((a_2, b_2) + (a_3, b_3)) \cdot (a_1, b_1) = (a_2, b_2) \cdot (a_1, b_1) + (a_3, b_3) \cdot (a_1, b_1).$$

We conclude our proposition. □

The additive identity $(0,0)$ is unique: If $(0',0'')$ is also an additive identity, then

$$(0,0) = (0',0'') + (0,0) = (0,0) + (0',0'') = (0',0'').$$

Similar argument also shows that the multiplicative identity $(1,0)$ is unique. In the context of abstract algebra, we sometimes called $(0,0)$ the *zero*, and called $(1,0)$ the *one*. We now define the "mysterious" element $\mathbf{i}$ rigorously.

DEFINITION 1.1.4. We define $\mathbf{i} := (0,1) \in \mathbb{C}$, and we call it the *imaginary unit*.

Obviously,

$$(a,0)\cdot(x,y) = (ax,ay), \tag{1.1.6}$$

and therefore in particular,

$$(a,0)\cdot(b,0) = (ab,0).$$

In addition, we have

$$(a,0)+(b,0) = (a+b,0).$$

Therefore, the mapping

$$\iota : \mathbb{R} \to \{(a,0) : a \in \mathbb{R}\}, \quad a \mapsto (a,0)$$

is a field isomorphism. Therefore, we somehow abuse the notation by simply writing

$$\mathbb{R} \equiv \{(a,0) : a \in \mathbb{R}\}, \quad 1 \equiv (1,0).$$

Since

$$(x,y) = (x,0)+(0,y) = (x,0)\cdot(1,0)+(y,0)\cdot(0,1),$$

then we see that:

LEMMA 1.1.5. *Each complex number $z = (x,y)$ can be written uniquely in the form $z = x + \mathbf{i}y$. The map*

$$\mathbb{R}\times\mathbb{R} \to \mathbb{C}, \quad (x,y) \mapsto x + y\mathbf{i}$$

is a bijection.

DEFINITION 1.1.6. If $z = x+\mathbf{i}y$, then we write $\mathfrak{Re}\, z := x$ (the real part of z) and $\mathfrak{Im}\, z := y$ (the imaginary part of z). Note that $\mathfrak{Im}\, z$ is a real number. We also define the conjugate $\overline{z}$ of z by $\overline{z} = x - \mathbf{i}y$.

It is useful to observe that

$$z + \overline{z} = 2x, \quad z - \overline{z} = 2\mathbf{i}y,$$

therefore,

$$\mathfrak{Re}\, z = \frac{1}{2}(z+\overline{z}), \quad \mathfrak{Im}\, z = \frac{1}{2\mathbf{i}}(z-\overline{z}) \equiv \frac{\mathbf{i}^{-1}}{2}(z-\overline{z}), \tag{1.1.7}$$

where $\mathbf{i}^{-1}$ is given by (1.1.5). From (1.1.3b), it is also useful to see that

$$\begin{aligned}&\overline{(a+\mathbf{i}b)\cdot(c+\mathbf{i}d)}\\ &\quad=\overline{(ac-bd)+\mathbf{i}(ad+bc)}\\ &\quad=(ac-bd)-\mathbf{i}(ad+bc)\\ &\quad=(ac-(-b)(-d))+\mathbf{i}(a(-d)+(-b)c)\\ &\quad=(a-\mathbf{i}b)\cdot(c-\mathbf{i}d)\\ &\quad=\overline{(a+\mathbf{i}b)}\cdot\overline{(c+\mathbf{i}d)},\end{aligned}$$

that is,

$$\overline{zw}=\overline{z}\cdot\overline{w}\quad\text{for all } z,w\in\mathbb{C}. \tag{1.1.8}$$

Therefore, one also can write (1.1.6) as

$$a(x+\mathbf{i}y)=ax+\mathbf{i}ay.$$

One can directly verify that

$$(\pm\mathbf{i})^2=\pm\mathbf{i}\cdot\pm\mathbf{i}=(0,\pm1)\cdot(0,\pm1)=(-1,0)\equiv-1.$$

This somehow suggests (1.1.2). At this moment, we first keep this question in mind, we will come back to answer this later.

1.2. Topological aspects of ℂ

We now discuss the topological aspects of the complex plane, in other words, we want to discuss how the open sets in ℂ looks like and define the continuous functions on ℂ. Here we also refer to the monograph [**Mun00**] for general abstract theory.

1.2.1. Sequences in ℂ. In this section, we shall see that there are many facts in calculus also holds true for complex numbers.

Definition 1.2.1. The *absolute value* (or *modulus*) $|z|$ of z, is defined by

$$|z|:=\sqrt{z\overline{z}}\equiv\sqrt{(\mathfrak{Re}\,z)^2+(\mathfrak{Im}\,z)^2}\equiv\|(\mathfrak{Re}\,z,\mathfrak{Im}\,z)\|_{\mathbb{R}^2},$$

which is just simply the Euclidean norm in $\mathbb{R}^2$.

It is not difficult to see the absolute homogeneity (i.e. $|rz|=|r||z|$ for all $r\in\mathbb{R}$) and positive definiteness of $|\cdot|$ (i.e. $|z|\geq 0$ and the equality holds if and only if $z=0$). To verify that $|\cdot|$ is a norm, we only need to verify the following:

Lemma 1.2.2 (Triangle inequality, subadditivity). $|z_1\pm z_2|\leq|z_1|+|z_2|$ *for all* $z_1,z_2\in\mathbb{C}$.

Proof. We now define the *inner product* on $\mathbb{R}^2\cong\mathbb{C}$ by

$$\langle z_1,z_2\rangle:=(\mathfrak{Re}\,z_1)(\mathfrak{Re}\,z_2)+(\mathfrak{Im}\,z_1)(\mathfrak{Im}\,z_2)\quad\text{for all } z_1,z_2\in\mathbb{C}.$$

One sees that $\langle z,z\rangle=(\mathfrak{Re}\,z)^2+(\mathfrak{Im}\,z)^2=|z|^2$. We also see that

$$\begin{aligned}|z_1\pm z_2|^2&=\langle z_1\pm z_2,z_1\pm z_2\rangle=|z_1|^2\pm2\langle z_1,z_2\rangle+|z_2|^2\\ (|z_1|+|z_2|)^2&=|z_1|^2+2|z_1||z_2|+|z_2|^2,\end{aligned}$$

therefore it is suffice to show the following Cauchy Schwartz inequality

$$\pm\langle z_1,z_2\rangle\leq|z_1||z_2|\quad\text{equivalently,}\quad|\langle z_1,z_2\rangle|\leq|z_1||z_2|.$$

In fact, we only need to prove the above inequality for the case both $z_1 \neq 0$ and $z_2 \neq 0$. In this case, by writing $w_1 := \frac{z_1}{|z_1|}$ and $w_2 := \frac{z_2}{|z_2|}$, we only need to prove

$$\pm\langle w_1, w_2\rangle \leq 1. \tag{1.2.1}$$

Since $|w_1| = |w_2| = 1$, then

$$0 \leq |w_1 \pm w_2|^2 = \langle w_1 \pm w_2, w_1 \pm w_2\rangle = 2 \pm 2\langle w_1, w_2\rangle,$$

which concludes (1.2.1). □

REMARK 1.2.3. We recall that $(\mathbb{C}, +, \cdot)$ forms a field, where $\cdot$ represents the complex multiplication. As a comparison, $(\mathbb{R}^2, +, \langle\cdot,\cdot\rangle)$ forms a ring, but not a field. Roughly speaking, we cannot define quotient for inner product, but we can define quotient for complex multiplication.

DEFINITION 1.2.4. The sequence $\{z_n\}_{n\in\mathbb{N}}$ *converges to* z *in* $\mathbb{C}$ if the sequence of real numbers $|z_n - z|$ converges to 0. Precisely, given any $\epsilon > 0$, there exists $N > 0$ such that $|z - z_n| < \epsilon$ for all $n \geq N$.

EXERCISE 1.2.5. Show that

$$\max\{|\mathfrak{Re}\, z|, |\mathfrak{Im}\, z|\} \leq |z| \leq |\mathfrak{Re}\, z| + |\mathfrak{Im}\, z|.$$

From this, one can easily see that $z_n \to z$ if and only if $\mathfrak{Re}\, z_n \to \mathfrak{Re}\, z$ and $\mathfrak{Im}\, z_n \to \mathfrak{Im}\, z$.

We also can rephrase the above definition in a more geometric terms:

$$\text{Given any } \epsilon > 0, \text{ there exists } N > 0 \text{ such that } z_n \in B_\epsilon(z) \text{ for all } n \geq N, \tag{1.2.2}$$

where $B_r(z)$ is the ball in $\mathbb{R}^2$ with radius r and centered at z. In the context of complex analysis, some authors refer $B_r(z)$ the *disk*.

While taking limit, we always need to check whether it exists or not, which is very inconvenient. For future convenience, here we recall a simple but nice concept, called the *limit supremum* and *limit infimum*. This should be already taught in calculus course. Here we follow [**Rud76**, Definition 3.16]. Given any sequence $\{a_n\} \subset \mathbb{R}$, we define

$$\begin{aligned}\limsup_{n\to\infty} a_n &:= \lim_{n\to\infty} \sup_{m\geq n} a_m \equiv \inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m,\\ \liminf_{n\to\infty} a_n &:= \lim_{n\to\infty} \inf_{m\geq n} a_m \equiv \sup_{n\in\mathbb{N}} \inf_{m\geq n} a_m.\end{aligned}$$

Unlike limit, both limit supremum and limit infimum always exist (because $\sup_{m\geq n} a_m$ and $\inf_{m\geq n} a_m$ are monotone), but may takes "values" $\pm\infty \notin \mathbb{R}$ (but only make sense for $\mathbb{R}$). It is clear that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n, \quad \liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n \quad \text{if } a_n \leq b_n \text{ for all } n \geq N \text{ for some } N > 0.$$

In addition, for a real-valued sequence $\{a_n\} \subset \mathbb{R}$, one has

$$\lim_{n\to\infty} a_n = a_\infty \in \mathbb{R} \iff \limsup_{n\to n} a_n = \liminf_{n\to n} a_n = a_\infty \in \mathbb{R}. \tag{1.2.3}$$

However, one has to be careful that, *we only have subadditivity (resp. superaddivity) property for limit supremum (resp. limit infimum)*:

$$\begin{cases} \limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \\ \liminf_{n\to\infty}(a_n+b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \end{cases} \quad \text{for } \{a_n\},\{b_n\} \subset \mathbb{R}, \tag{1.2.4}$$

holds whenever the right hand side is not $\infty-\infty$ or $-\infty+\infty$. **For the case when $\lim_{n\to\infty} b_n$ exists and finite**, by writing $a_n = (a_n+b_n)+(-b_n)$, using (1.2.4) we obtain

$$\begin{cases} \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n+b_n) - \lim_{n\to\infty} b_n \\ \liminf_{n\to\infty} a_n \geq \liminf_{n\to\infty}(a_n+b_n) - \lim_{n\to\infty} b_n \end{cases}$$

which implies

$$\begin{cases} \limsup_{n\to\infty} a_n + \lim_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n+b_n), \\ \liminf_{n\to\infty} a_n + \lim_{n\to\infty} b_n \geq \liminf_{n\to\infty}(a_n+b_n). \end{cases}$$

Combining this with (1.2.4), we reach

$$\begin{cases} \limsup_{n\to\infty}(a_n+b_n) = \limsup_{n\to\infty} a_n + \lim_{n\to\infty} b_n \\ \liminf_{n\to\infty}(a_n+b_n) = \liminf_{n\to\infty} a_n + \lim_{n\to\infty} b_n \end{cases} \quad \text{when } \lim_{n\to\infty} b_n \text{ exists and finite.} \tag{1.2.5}$$

If $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n$ exists which converges to some $b \geq 0$, by writing $a_nb_n = a_nb + a_n(b_n-b)$ and using (1.2.5), one sees that

$$\begin{cases} \limsup_{n\to\infty}(a_nb_n) = \limsup_{n\to\infty}(a_nb) \overset{(\because b\geq 0)}{\equiv} \left(\limsup_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right), \\ \liminf_{n\to\infty}(a_nb_n) = \liminf_{n\to\infty}(a_nb) \overset{(\because b\geq 0)}{\equiv} \left(\liminf_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right). \end{cases} \tag{1.2.6}$$

If we choose trivial sequence $b_n = b \geq 0$ for all n, then we reach

$$\limsup_{n\to\infty}(ba_n) = b \limsup_{n\to\infty} a_n \quad \text{for } b \geq 0. \tag{1.2.7}$$

However, one should be aware that when $b \geq 0$, we have

$$\limsup_{n\to\infty}(ba_n) = -\liminf_{n\to\infty}(|b|a_n) = -|b| \liminf_{n\to\infty} a_n = b \liminf_{n\to\infty} a_n \quad \text{for } b \leq 0.$$

EXERCISE 1.2.6. Compute $\limsup_{n\to\infty}(a_nb_n)$ and $\liminf_{n\to\infty}(a_nb_n)$ when $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n$ exists which converges to some $b \leq 0$.

If both $\{a_n\}$ and $\{b_n\}$ are non-negative, one also has

$$\begin{cases} \limsup_{n\to\infty}(a_nb_n) \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right) \\ \liminf_{n\to\infty}(a_nb_n) \geq \left(\liminf_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right) \end{cases} \quad \text{for } \textbf{non-negative } \{a_n\},\{b_n\} \tag{1.2.8}$$

holds whenever the right hand side is not $0\cdot\infty$ or $\infty\cdot 0$. From (1.2.3) we have the following:

LEMMA 1.2.7. *$z_n \to z \in \mathbb{C}$ if and only if $\limsup_{n\to\infty} |z_n - z| = 0$.*

This simple observation can simplify the proofs. We can always take limit supremum in the proof, which may simplify the proof in the future. One only need to be careful about (1.2.4).

DEFINITION 1.2.8. The sequence $\{z_n\}_{n\in\mathbb{N}}$ is called a *Cauchy sequence* in $\mathbb{C}$ if, given any $\epsilon > 0$, there exists $N > 0$ such that $|z_n - z_m| < \epsilon$ for all $n, m \geq N$.

LEMMA 1.2.9. *The complex field* $(\mathbb{C}, |\cdot|)$ *is complete, that is, the sequence* $\{z_n\}$ *converges if and only if* $\{z_n\}$ *is a Cauchy sequence.*

PROOF. We first assume that the sequence $\{z_n\}$ converges to some limit z. By using the triangle inequality in Lemma 1.2.2, one has

$$|z_n - z_m| \leq |z_n - z| + |z - z_m|,$$

which immediately shows that $\{z_n\}$ is Cauchy. Conversely, suppose that $\{z_n\}$ is a Cauchy sequence. From Definition 1.2.1 it is easy to see that

$$\begin{aligned} |\mathfrak{Re}\, z_n - \mathfrak{Re}\, z_m| &= |\mathfrak{Re}\,(z_n - z_m)| \leq |z_n - z_m|, \\ |\mathfrak{Im}\, z_n - \mathfrak{Im}\, z_m| &= |\mathfrak{Im}\,(z_n - z_m)| \leq |z_n - z_m|, \end{aligned}$$

so that both $\{\mathfrak{Re}\, z_n\}$ and $\{\mathfrak{Im}\, z_n\}$ form Cauchy sequence in $\mathbb{R}$, therefore there exist $a, b \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \mathfrak{Re}\, z_n = a, \quad \lim_{n\to\infty} \mathfrak{Im}\, z_n = b.$$

We define $z := a + b\mathbf{i}$, and from Exercise 1.2.5 and (1.2.4), one sees that

$$\begin{aligned} \limsup_{n\to\infty} |z_n - z| &\leq \limsup_{n\to\infty} (|\mathfrak{Re}\,(z_n - z)| + |\mathfrak{Im}\,(z_n - z)|) \\ &\leq \limsup_{n\to\infty} |\mathfrak{Re}\,(z_n - z)| + \limsup_{n\to\infty} |\mathfrak{Im}\,(z_n - z)| \\ &= \limsup_{n\to\infty} |\mathfrak{Re}\, z_n - a| + \limsup_{n\to\infty} |\mathfrak{Im}\, z_n - b| = 0, \end{aligned} \tag{1.2.9}$$

which conclude our lemma. □

DEFINITION 1.2.10. We now given a sequence $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$, and we define its partial sum

$$s_n := \sum_{k=1}^{n} z_k.$$

An infinite series $\sum_{k=1}^{\infty} z_k$ is said to *converge* in $\mathbb{C}$ if s_n converges in $\mathbb{C}$.

The following basic properties can be proved using same ideas as in calculus:

(1) If $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} w_k$ are converge in $\mathbb{C}$, then $\sum_{k=1}^{\infty} (z_k \pm w_k)$ are converge in $\mathbb{C}$.
(2) If $\sum_{k=1}^{\infty} z_k$ is converges in $\mathbb{C}$, then $z_k \to 0 \in \mathbb{C}$.
(3) If $\sum_{k=1}^{\infty} |z_k|$ converges in $\mathbb{R}$, then $\sum_{k=1}^{\infty} z_k$ converges in $\mathbb{C}$ (this can be easily proved using triangle inequality in Lemma 1.2.2).

DEFINITION 1.2.11. If $\sum_{k=1}^{\infty} |z_k|$ converges in $\mathbb{R}$, then we say that $\sum_{k=1}^{\infty} z_k$ converges in $\mathbb{C}$ *absolutely*. Otherwise, we call the convergence is *conditionally*.

1.2.2. Open sets in complex plane $\mathbb{C}$.

DEFINITION 1.2.12. Let Ω be a set in $\mathbb{C}$. We say that Ω is open in $\mathbb{C}$ if, given any $z \in \Omega$, there exists a $\epsilon > 0$ such that $B_\epsilon(z) \subset \Omega$.

This means that the open sets in $\mathbb{C}$ is exactly same as in $\mathbb{R}^2$. Therefore we can borrow a lot of topological terminology from $\mathbb{R}^2$:

(1) An open set Ω contained z sometimes called the *neighborhood* of z.

(2) A set A is *topological closed* in $\mathbb{C}$ if its complement $A^{\complement} := \mathbb{C} \setminus A$ is open in $\mathbb{C}$. In this case, A is closed in $\mathbb{C}$ if and only any Cauchy sequence $\{z_n\} \subset A$ converges to a limit $z \in A$.
(3) The *boundary* ∂S of a set S is defined as: $x \in \partial S$ if and only if $B_\epsilon(x) \cap S \neq \emptyset$ and $B_\epsilon(x) \cap S^{\complement} \neq \emptyset$ for all $\epsilon > 0$.
(4) The *closure* $\overline{S}$ of a set S is defined by $\overline{S} := S \cup \partial S$.
(5) Sometimes we called the boundary $\partial B_R(z)$ of a ball $B_R(z)$ the *circle*.
(6) A set S is *bounded* if $S \subset B_R \equiv B_R(0)$ for some $R > 0$.

Definition 1.2.13. A set S is called *compact* in $\mathbb{C}$ if the following holds:

$$\begin{aligned} &S \subset \bigcup_{\alpha\in\Lambda} \mathscr{O}_\alpha \text{ for some collection of open sets } \{\mathscr{O}_\alpha\}_{\alpha\in\Lambda} \\ &\implies S \subset \bigcup_{\alpha\in\Lambda'} \mathscr{O}_\alpha \text{ for some } \Lambda' \subset \Lambda \text{ which is finite.} \end{aligned}$$

In fact, we have the Heine-Borel theorem: S is compact in $\mathbb{C}$ if and only if S is topological closed and bounded. Using Bolzano-Weierstrass theorem, we also see that S is compact in $\mathbb{C}$ if and only if any sequence in S must have a subsequence which is converges in S.

Definition 1.2.14. Let S be any set in $\mathbb{C}$. A subset $S_0 \subset S$ is said to be *relative open* in S if there exists an open set $\Omega \subset \mathbb{C}$ such that $S_0 = S \cap \Omega$. Similarly, a subset $S_1 \subset S$ is said to be relative topological closed in S if there exists a topological closed set $F \subset \mathbb{C}$ such that $S_1 = S \cap F$. A set S is said to be *connected* if the following holds:

$$\begin{aligned} &\text{if } S_0 \subset S \text{ is both relative open and relative topological closed in } S \\ &\text{then either } S_0 = \emptyset \text{ or } S_0 = S. \end{aligned} \tag{1.2.10}$$

Remark 1.2.15 (Relative open sets in open sets). If S is an open set (resp. topological closed set) in $\mathbb{C}$ and $S_0 \subset S$, then S_0 is open (resp. topological closed) in $\mathbb{C}$ if and only if S_0 is relative open (resp. relative tolopogical closed) in S. This can be easily see by the trivial set inclusion $S_0 = S \cap S_0$.

It is make sense to say that a set S is said to be disconnected if (1.2.10) does not hold. This means that there exists $\emptyset \neq S_0 \subsetneq S$

$$\begin{aligned} &\text{there exists } \emptyset \neq S_0 \subsetneq S \text{ such that} \\ &S_0 \text{ is both relative open and relative topological closed in } S. \end{aligned}$$

In this case, if we define $S_1 := S \setminus S_0$, it is easy to see that $\emptyset \neq S_1 \subsetneq S_0$ is also both relative open and relative topological closed in S. Therefore one see that S_0 and S_1 are both disjoint (open) components of S.

Definition 1.2.16. We denote $[z_1, z_2]$ the line segment with endpoints z_1 and z_2. A polygonal line is a finite union of line segments of the form $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$.

Lemma 1.2.17. *Let Ω be an open set in $\mathbb{C}$. Then Ω is connected if and only if for any $a, b \in \Omega$ there exists a polygonal line in Ω connecting a and b.*

Remark 1.2.18. Sometimes we also called an open connected set a *region* or *domain*.

Proof of Lemma 1.2.17. "$\Rightarrow$" Let $a \in \Omega$ and let

$$A := \left\{ x \in \Omega \,\middle|\, \text{there exists a polygonal line connecting } a \text{ and } x \right\}.$$

It is clear that $a \in A$, which shows that $A \neq \emptyset$.

Given any $x \in A \subset \Omega$, since Ω is open, then there exists $\epsilon = \epsilon(x) > 0$ such that $B_\epsilon(x) \subset \Omega$. Clearly any point in $B_\epsilon(x)$ can be connected to x by using a straight line, then any point in $B_\epsilon(x)$ can be connected to a by a polygonal line. In other words, $B_\epsilon(x) \subset A$. By arbitrariness of $x \in A$, we conclude that A is open in $\mathbb{C}$, and hence also relative open in Ω.

Similar argument shows that $\Omega \setminus A$ is also relative open in Ω. This shows that A is relative topological closed in Ω. Since $A \neq \emptyset$, then $A = \Omega$.

"$\Leftarrow$" Let $\emptyset \neq A \subset \Omega$ be a set such that it is both relative open and relative topological closed in Ω. Suppose the contrary, that $A \neq \Omega$, i.e. $\Omega \setminus A \neq \emptyset$. Choose $a \in A$ and $b \in \Omega \setminus A$. By assumption, one can find a polygonal line connecting a and b, says $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$ with $z_0 = a$ and $z_n = b$. We define a continuous function f on $[0, n]$ by

$$f(t) = z_j + (t - j)(z_{j+1} - z_j) \text{ when } t \in [j, j+1] \text{ for } j = 0, 1, \cdots, n-1.$$

We now define the sets (called the preimage, this is just a notation, does not mean f is invertible)

$$f^{-1}(A) := \left\{ \, x \in \Omega \;\middle|\; f(x) \in A \, \right\}, \quad f^{-1}(\Omega \setminus A) := \left\{ \, x \in \Omega \;\middle|\; f(x) \in \Omega \setminus A \, \right\}.$$

Since both A and $\Omega \setminus A$ are open (in $\mathbb{C}$ if and only if relative to Ω), then both $f^{-1}(A)$ and $f^{-1}(S \setminus A)$ are relative open in $[0, n]$. This is a special case of a general topological fact, but here we give a simple argument to show that both $f^{-1}(A)$ and $f^{-1}(S \setminus A)$ are relative open in $[0, n]$. We only show $f^{-1}(A)$ is relative open in $[0, n]$, since the same thing can be done for $f^{-1}(S \setminus A)$. Let $x_0 \in f^{-1}(A)$, i.e. $f(x_0) \in A$.

Case 1: $x_0 \neq 0$ and $x_0 \neq n$. Since A is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subset A$. By continuity of f at x_0, there exists $\delta > 0$ such that

$$\overbrace{|y - x_0| < \delta}^{\iff y \in B_\delta(x_0)} \implies \overbrace{|f(y) - f(x_0)| < \epsilon}^{\iff f(y) \in B_\epsilon(f(x_0))}.$$

Hence we see that

$$\overbrace{|y - x_0| < \delta}^{\iff y \in B_\delta(x_0)} \implies \overbrace{f(y) \subset A}^{\iff y \in f^{-1}(A)}$$

this meas that $B_\delta(x_0) \subset f^{-1}(A)$.

Case 2: $x_0 = 0$ (similar treatment for $x_0 = n$). In this case, the continuity of f at $x_0 = 0$ means there exists $\delta > 0$ (without loss of generality, we may choose $\delta < n$) such that

$$\overbrace{0 \leq y \equiv y - x_0 < \delta}^{\iff y \in B_\delta(x_0) \cap [0,n]} \implies \overbrace{|f(y) - f(x_0)| < \epsilon}^{\iff f(y) \in B_\epsilon(f(x_0))}.$$

Hence we see that

$$\overbrace{0 \leq y \equiv y - x_0 < \delta}^{\iff y \in B_\delta(x_0) \cap [0,n]} \implies \overbrace{f(y) \subset A}^{\iff y \in f^{-1}(A)}.$$

This means that $B_\delta(x_0) \cap [0, n] \subset f^{-1}(A)$.

Combining these two cases, we now conclude that

given any $x \in f^{-1}(A)$, there exists $\delta = \delta(x) > 0$ such that $B_\delta(x) \cap [0, 1] \subset f^{-1}(A)$.

This means that $f^{-1}(A)$ is relative open in $[0,n]$, because

$$f^{-1}(A) = [0,1] \cap \overbrace{\bigcup_{x \in f^{-1}(A)} B_{\delta(x)}(x)}^{\text{open in } \mathbb{C}}.$$

Similar arguments also show that $f^{-1}(S \setminus A)$ is relative open in $[0,n]$, and hence $f^{-1}(A)$ is relative topological closed in $[0,n]$. Since the interval $[0,n]$ is connected and $f^{-1}(A) \neq \emptyset$, then $f^{-1}(A) = [0,n]$ and hence $f^{-1}(S \setminus A) = \emptyset$, which is a contradiction. This means that the assumption $A \neq \Omega$ in the contradiction argument does not hold. Hence we conclude that $A = \Omega$. $\square$

REMARK 1.2.19. The above exhibits a standard argument when dealing with open connected set:

(1) First show that the target set A (i.e. the set of the property which we wish to show) is nonempty.
(2) Show that A is relative open.
(3) Show that $\Omega \setminus A$ is relative open.

To show an open set is connected, one of course can try to construct a continuous path

In my opinion, even though Lemma 1.2.17 gives a quite easy understanding, but Mathematically sometimes this characterization is not convenient to manipulate. Personally I prefer the definition (1.2.10): Even though it is abstract, but this is quite convenient to manipulate in Mathematical proof.

LEMMA 1.2.20. *$z_n \to z$ if and only if: Given any open set $\Omega \ni z$, there exists $N > 0$ such that $z_n \in \Omega$ for all $n \geq N$.*

PROOF. We first suppose that $z_n \to z$. Given any open set $\Omega \ni z$, by definition there exists $\epsilon > 0$ such that

$$B_\epsilon(z) \subset \Omega.$$

By using (1.2.2), there exists $N > 0$ such that $z_n \in B_\epsilon(z) \subset \Omega$ for all $n \geq N$, which complete our proof. The converse is trivial by choosing $\Omega = B_\epsilon(z)$ for arbitrary $\epsilon > 0$. $\square$

1.2.3. Continuous functions on $\mathbb{C}$.

DEFINITION 1.2.21. Let $z \in \mathbb{C}$ and let Ω be an open neighborhood of z. We say that function $f : \Omega \to \mathbb{C}$ is continuous at z if

$$z_n \to z \in \mathbb{C} \implies f(z_n) \to f(z) \in \mathbb{C}.$$

Alternatively, given any $\epsilon > 0$, there exists $\delta > 0$, which depends on z, such that

$$|f(z) - f(y)| < \epsilon \text{ for all } |z - y| < \delta. \tag{1.2.11}$$

In other words, $f(y) \in B_\epsilon(f(z))$ for all $y \in B_\delta(x)$. We say that f is continuous on Ω, denoted by $f \in C(\Omega)$, if f is continuous at all point $z \in \Omega$.

REMARK 1.2.22. If one can find δ in (1.2.11) which is independent of $z \in \Omega$, then one call such function is *uniformly continuous*. In this case, it is also convenient to write (1.2.11) as

$$\sup_{z,y \in \Omega, |z-y| < \delta} |f(z) - f(y)| < \epsilon.$$

This notation emphasized that δ is independent of both y and z.

If we split f into its real and imaginary parts

$$f(z) = u(x,y) + \mathbf{i}v(x,y) \quad \text{for } z = x + \mathbf{i}y \in \Omega$$

it is clear that f is continuous at $z = x + y\mathbf{i}$ if and only if both u and v continuous at (x,y).

DEFINITION 1.2.23. We say that $f \in C^m$ if and only if both $u, v \in C^m$, i.e. have continuous partial derivatives of the m^{th} order.

DEFINITION 1.2.24. A sequence of functions $\{f_n\}$ is said to be converge to f uniformly in Ω, if for each $\epsilon > 0$ there is an $N > 0$, which independent of $z \in \Omega$, such that

$$n \geq N \implies |f_n(z) - f(z)| < \epsilon \text{ for all } z \in \Omega. \tag{1.2.12}$$

We now define the sup-norm on Ω by

$$\|g\|_{L^\infty(\Omega)} := \sup_{z\in\Omega} |g(z)| \quad \text{for all } g \in C(\Omega)$$

By using this notations, we see that is equivalent to

$$\|f_n - f\|_{L^\infty(\Omega)} \equiv \sup_{z\in\Omega} |f_n(z) - f(z)| < \epsilon \quad \text{for all } n \geq N. \tag{1.2.13}$$

LEMMA 1.2.25. *Let Ω be an open set in $\mathbb{C}$. Then f_n converges to f uniformly in Ω if and only if* $\limsup_{n\to\infty} \|f_n - f\|_{L^\infty(\Omega)} = 0$.

Therefore, we also can say that $f_n \to f$ in $L^\infty(\Omega)$-sense. Sometimes we also refer f the uniform limit of f. It is well-known (see e.g. [**Rud76**]) that the uniform limit of real-valued continuous function is continuous. By using the triangle inequality of $\|\cdot\|_{L^\infty(\Omega)}$, which can be easily proved using Lemma 1.2.2, one can easily obtain the following lemma.

LEMMA 1.2.26. *Let Ω be an open set in $\mathbb{C}$ and let $\{f_n\} \subset C(\Omega)$. If f_n converges to f uniformly in Ω, then $f \in C(\Omega)$.*

COROLLARY 1.2.27 (Weierstrass M-test). *Let Ω be an open set in $\mathbb{C}$ and let $\{f_n\} \subset C(\Omega)$. If $\|f_k\|_{L^\infty(\Omega)} \leq M_k$ and $\sum_{k=1}^\infty M_k$ converges in $\mathbb{R}$, then $\sum_{k=1}^\infty f_k(z)$ converges to a continuous function uniformly in Ω.*

PROOF. It is easy to see that $f(z) = \sum_{k=1}^\infty f_k(z)$ pointwisely. Moreover, we see that

$$\limsup_{n\to\infty} \left\| f - \sum_{k=1}^n f_k \right\|_{L^\infty(\Omega)} = \limsup_{n\to\infty} \left\| \sum_{k=n+1}^\infty f_k \right\|_{L^\infty(\Omega)} \leq \limsup_{n\to\infty} \sum_{k=n+1}^\infty M_k = 0,$$

which concludes our corollary. □

REMARK 1.2.28 (A general trick). Here is a suggested standard procedure of proving uniform convergence: First prove pointwise convergence to make sure the existence of limit function (candidate), and then verify the convergence is uniform. This procedure is based on the fact that the uniform limit is necessarily also a pointwise limit.

CHAPTER 2

Differentiation

2.1. Complex derivative and Cauchy-Riemann equation

Inspired by calculus, it is not surprising to introduce the following definition.

Definition 2.1.1. A complex-valued function f, defined in a neighborhood of z, is said to be (complex) *differentiable* at z if

$$\lim_{\mathbb{C}\ni h\to 0} \frac{f(z+h)-f(z)}{h} \text{ exists.}$$

In this case, the limit is denoted by $f'(z)$ or $\partial_z f(z)$ or $\frac{\partial}{\partial z} f(z)$ or $\frac{\mathrm{d}}{\mathrm{d}z} f(z)$. Let Ω be an open set in $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ which is differentiable at every point Ω is also called (complex) *analytic* or *holomorphic* in Ω. A function $f : \mathbb{C} \to \mathbb{C}$ which is differentiable at every point $\mathbb{C}$ is also called *entire*.

Remark 2.1.2. It is important to note that in the above definition, h is not necessarily real.

Remark 2.1.3. Let Ω be an open set in $\mathbb{C}$. Some authors call a function $f : \Omega \to \mathbb{C}$ is called analytic at a point $a \in \Omega$ if there exists an open neighborhood $U \subset \Omega$ of a such that f is analytic in U. Personally, I would prefer to say

$$\text{such function } f \text{ is analytic } \textbf{near } a \in \Omega \quad \text{(rather than "at").} \tag{2.1.1}$$

Throughout this course, we shall use the terminology (2.1.1) to avoid confusion.

Exercise 2.1.4. Show that the function $f(z) = z\overline{z}$ is differentiable at $z = 0$, but not analytic near $z = 0$.

This exercise reminds us to be carefully while stating the terms “at” and “near”.

Lemma 2.1.5. *If f and g are both differentiable at z, then so are $h_1 = f + g$ and $h_2 = fg$. If $g'(z) \neq 0$, then $h_3 = f/g$ also differentiable at z. In the respective cases,*

$$\begin{aligned} h_1'(z) &= f'(z) + g'(z), \\ h_2'(z) &= f'(z)g(z) + f(z)g'(z), \\ h_3'(z) &= \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}. \end{aligned}$$

Example 2.1.6. If $P(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_N z^N$ for some complex numbers $\alpha_0, \cdots, \alpha_N$, then P is differentiable at all points z and $P'(z) = \alpha_1 + 2\alpha_2 z + \cdots + N\alpha_N z^{N-1}$.

Exercise 2.1.7. Prove Lemma 2.1.5 and verify Example 2.1.6.

Lemma 2.1.8. *If $f = u + \mathbf{i}v$ is differentiable at $z = x + \mathbf{i}y$, then the partial derivatives $\partial_x f$ and $\partial_y f$ of f both exist, and they satisfy the Cauchy-Riemann equation $\partial_y f = \mathbf{i}\partial_x f$.*

PROOF. The existence of $\partial_x f$ and $\partial_y f$ can be easily seen from the identities

$$\lim_{\mathbb{R}\ni h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{\mathbb{R}\ni h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \partial_x f(x,y),$$
$$\lim_{\mathbb{R}\ni h\to 0} \frac{f(z+\mathbf{i}h)-f(z)}{\mathbf{i}h} = \lim_{\mathbb{R}\ni h\to 0} \frac{f(x,y+h)-f(x,y)}{\mathbf{i}h} = \frac{1}{\mathbf{i}}\partial_y f(x,y).$$

Since f is differentiable at z, then the above identities must be identical, which conclude our lemma. □

The converse of the above lemma does not hold true: There exist functions which are not differentiable at a point despite the fact that the partial derivatives exist and satisfy the Cauchy-Riemann equations here.

EXAMPLE 2.1.9. We consider

$$f(z) = f(x,y) = \begin{cases} \frac{xy(x+\mathbf{i}y)}{x^2+y^2} & , z \neq 0, \\ 0 & , z = 0. \end{cases}$$

Since $f = 0$ on both axes $x = 0$ and $y = 0$, so that $\partial_x f(0,0) = \partial_y f(0,0) = 0$ (and hence satisfies the Cauchy-Riemann equation). However, for each $\alpha \in \mathbb{R}$, one sees that

$$\lim_{z\to 0, y=\alpha x} \frac{f(z)-f(0)}{z} = \lim_{z=x+\mathbf{i}\alpha x\to 0} \frac{x(\alpha x)(x+\mathbf{i}\alpha x)}{x^2+(\alpha x)^2} = \frac{\alpha}{1+\alpha^2}.$$

This shows that $\partial_z f(0,0)$ does not exist. Suppose the contrary, that $\partial_z f(0,0)$ exists, then

$$\partial_z f(0,0) = \lim_{\mathbb{C}\ni z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0, y=\alpha x} \frac{f(z)-f(0)}{z} = \frac{\alpha}{1+\alpha^2} \quad \text{for all } \alpha \in \mathbb{R},$$

which is a contradiction since $\partial_z f(0,0)$ is independent of α.

However, it is worth to mention and proof that the equivalence holds when f is sufficiently regular:

THEOREM 2.1.10. *Suppose that $f \in C^1$ in a neighborhood of $z = x + \mathbf{i}y$ (sometimes we simply say $f \in C^1$ near z), that is, $\partial_x f$ and $\partial_y f$ are continuous in a neighborhood of z. We have the following equivalence:*

f satisfies the Cauchy-Riemann equation $\partial_y f = \mathbf{i}\partial_x f$ at z $\iff$ f is differentiable at z.

REMARK 2.1.11. If we write $f = u + \mathbf{i}v$, the Cauchy-Riemann equation can be written as

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

From this, we see that

$$\Delta u := \partial_x^2 u + \partial_y^2 u = \partial_x\partial_y v - \partial_y\partial_x v = 0,$$
$$\Delta v := \partial_x^2 v + \partial_y^2 v = -\partial_x\partial_y u + \partial_y\partial_x u = 0,$$

in other words, both u and v are harmonic.

PROOF OF THEOREM 2.1.10. The implication "⇐" was proved by Lemma 2.1.8. We only need to prove the implication "⇒".

We write $h = h_1 + \mathbf{i}h_2$. By using mean value theorem (for real functions of a real variable), one sees that

$$\begin{aligned}
&\frac{\mathfrak{Re}\, f(z+h) - \mathfrak{Re}\, f(z)}{h} = \frac{\mathfrak{Re}\, f(x+h_1, y+h_2) - \mathfrak{Re}\, f(z)}{h_1 + \mathbf{i}h_2} \\
&\quad = \frac{\mathfrak{Re}\, f(x+h_1, y+h_2) - \mathfrak{Re}\, f(x+h_1, y)}{h_1 + \mathbf{i}h_2} + \frac{\mathfrak{Re}\, f(x+h_1, y) - \mathfrak{Re}\, f(x, y)}{h_1 + \mathbf{i}h_2} \\
&\quad = \frac{h_2}{h_1 + \mathbf{i}h_2} \partial_y \mathfrak{Re}\, f(x+h_1, y+\eta_2) + \frac{h_1}{h_1 + \mathbf{i}h_2} \partial_x \mathfrak{Re}\, f(x+\eta_1, y) \\
&\quad = \frac{\mathbf{i}h_2}{h_1 + \mathbf{i}h_2} \frac{1}{\mathbf{i}} \partial_y \mathfrak{Re}\, f(x+h_1, y+\eta_2) + \frac{h_1}{h_1 + \mathbf{i}h_2} \partial_x \mathfrak{Re}\, f(x+\eta_1, y)
\end{aligned}$$

for some $\eta_1 \leq |h_1|$ and $\eta_2 \leq |h_2|$. Using the exactly same arguments, one also see that

$$\begin{aligned}
&\frac{\mathfrak{Im}\, f(z+h) - \mathfrak{Im}\, f(z)}{h} \\
&\quad = \frac{\mathbf{i}h_2}{h_1 + \mathbf{i}h_2} \frac{1}{\mathbf{i}} \partial_y \mathfrak{Im}\, f(x+h_1, y+\eta_4) + \frac{h_1}{h_1 + \mathbf{i}h_2} \partial_x \mathfrak{Im}\, f(x+\eta_3, y)
\end{aligned}$$

for some $\eta_3 \leq |h_1|$ and $\eta_4 \leq |h_2|$. Therefore, one has

$$\begin{aligned}
&\frac{f(z+h) - f(z)}{h} - \partial_x f(z) \\
&\quad = \frac{\mathbf{i}h_2}{h_1 + \mathbf{i}h_2} \left(\frac{1}{\mathbf{i}} \partial_y \left(\mathfrak{Re} f(x+h_1, y+\eta_2) + \mathbf{i}\mathfrak{Im}\, f(x+h_1, y+\eta_4) \right) - \partial_x f(x, y) \right) \\
&\qquad + \frac{h_1}{h_1 + \mathbf{i}h_2} \left(\partial_x \left(\mathfrak{Re}\, f(x+\eta_1, y) + \mathbf{i}\mathfrak{Im}\, f(x+\eta_3, y) \right) - \partial_x f(x, y) \right).
\end{aligned}$$

By using the Cauchy-Riemann equation, we can write the above equation as

$$\begin{aligned}
&\frac{f(z+h)-f(z)}{h} - \partial_x f(z) \\
&\quad = \frac{\mathbf{i}h_2}{h_1+\mathbf{i}h_2}\left(\frac{1}{\mathbf{i}}\partial_y\left(\mathfrak{Re} f(x+h_1,y+\eta_2)+\mathbf{i}\mathfrak{Im}\, f(x+h_1,y+\eta_4)\right) - \frac{1}{\mathbf{i}}\partial_y f(x,y)\right) \\
&\qquad + \frac{h_1}{h_1+\mathbf{i}h_2}\left(\partial_x\left(\mathfrak{Re}\, f(x+\eta_1,y)+\mathbf{i}\mathfrak{Im}\, f(x+\eta_3,y)\right) - \partial_x f(x,y)\right) \\
&\quad = \frac{h_2}{h_1+\mathbf{i}h_2}\left(\partial_y\left(\mathfrak{Re} f(x+h_1,y+\eta_2)+\mathbf{i}\mathfrak{Im}\, f(x+h_1,y+\eta_4)\right) - \partial_y f(x,y)\right) \\
&\qquad + \frac{h_1}{h_1+\mathbf{i}h_2}\left(\partial_x\left(\mathfrak{Re}\, f(x+\eta_1,y)+\mathbf{i}\mathfrak{Im}\, f(x+\eta_3,y)\right) - \partial_x f(x,y)\right) \\
&\quad = \frac{h_2}{h_1+\mathbf{i}h_2}\left(\partial_y\mathfrak{Re} f(x+h_1,y+\eta_2) - \partial_y\mathfrak{Re}\, f(x,y)\right) \\
&\qquad + \frac{\mathbf{i}h_2}{h_1+\mathbf{i}h_2}\left(\partial_y\mathfrak{Im}\, f(x+h_1,y+\eta_4) - \partial_y\mathfrak{Im}\, f(x,y)\right) \\
&\qquad + \frac{h_1}{h_1+\mathbf{i}h_2}\left(\partial_x\mathfrak{Re} f(x+\eta_1,y) - \partial_x\mathfrak{Re}\, f(x,y)\right) \\
&\qquad + \frac{\mathbf{i}h_1}{h_1+\mathbf{i}h_2}\left(\partial_x\mathfrak{Im}\, f(x+\eta_3,y) - \partial_x\mathfrak{Im}\, f(x,y)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\limsup_{\mathbb{C}\ni h\to 0}\left|\frac{f(z+h)-f(z)}{h} - \partial_x f(z)\right| \\
&\quad \le \limsup_{\mathbb{C}\ni h\to 0}\left|\partial_y\mathfrak{Re} f(x+h_1,y+\eta_2) - \partial_y\mathfrak{Re}\, f(x,y)\right| \\
&\qquad + \limsup_{\mathbb{C}\ni h\to 0}\left|\partial_y\mathfrak{Im}\, f(x+h_1,y+\eta_4 - \partial_y\mathfrak{Im}\, f(x,y)\right| \\
&\qquad + \limsup_{\mathbb{C}\ni h\to 0}\left|\partial_x\mathfrak{Re} f(x+\eta_1,y) - \partial_x\mathfrak{Re}\, f(x,y)\right| \\
&\qquad + \limsup_{\mathbb{C}\ni h\to 0}\left|\partial_x\mathfrak{Im}\, f(x+\eta_3,y) - \partial_x\mathfrak{Im}\, f(x,y)\right| \\
&\quad = 0,
\end{aligned}$$

which complete our proof with $\partial_z f = \partial_x f$. □

REMARK 2.1.12. Suppose that all assumptions in Theorem 2.1.10 hold near $z \in \mathbb{C}$. Let f be a complex-valued function which is analytic at z. By using the Cauchy-Riemann equation and $\partial_z f = \partial_x f$, one see that

$$\partial_z f = \frac{1}{2}(\partial_x f - \mathbf{i}\partial_y f). \tag{2.1.2a}$$

We now define the operator $\partial_{\overline{z}}$ on C^1 function by

$$\partial_{\overline{z}} f := \frac{1}{2}(\partial_x f + \mathbf{i}\partial_y f). \tag{2.1.2b}$$

By introducing this notation, one sees that Theorem 2.1.10 can be restated as

$$\begin{gathered} f \text{ is differentiable at } z \iff \partial_{\overline{z}} f = 0 \text{ at } z \\ \text{(assuming that all assumptions in Theorem 2.1.10 hold)} \end{gathered} \tag{2.1.3}$$

In particular, the operators (2.1.2a) and (2.1.2b) are called the *Wirtinger operators*.

REMARK 2.1.13. Wirtinger operators can be defined in terms of weak derivatives (even distributional derivatives), and it interesting to mention that the quasiconformal mapping is related to the *Beltrami equation*:

$$\partial_{\overline{z}} f = \mu \partial_z f \quad \text{with} \quad \|\mu\|_{L^\infty} \leq c < 1.$$

When $\mu \equiv 0$, this reduces to (2.1.3) (Note: We called a mapping is conformal if it is holomorphic and injective, therefore the term "quasiconformal" make sense). For more details about the quasiconformal mapping and Beltrami equation, one can refer to the monograph [**AIM09**].

Warning: If $\partial_{\overline{z}} f \neq 0$ (i.e. does not satisfy Cauchy-Riemann equation), the function $\partial_z f$ in (2.1.2a) is not equivalent to the one we introduced in Definition 2.1.1.

Warning: Even though (2.1.3) suggests that analytic function must not contained $\overline{z}$, to show a function is analytic or not, we still have to verify the definition carefully, see Exercise 2.1.4.

Warning: Always remember to check the assumptions in Theorem 2.1.10.

EXAMPLE 2.1.14. Any complex-valued polynomial P takes the form $P = \sum_{n=0}^{N} Q_n$ for some $N \in \mathbb{Z}_{\geq 0}$ with

$$\begin{aligned} Q_n(z) = Q_n(x,y) &= \sum_{k=0}^{n} C_{n,k} x^{n-k} y^k \\ &= C_{n,0} x^n + C_{n,1} x^{n-1} y + C_{n,2} x^{n-2} y^2 + \cdots + C_{n,n} y^n \end{aligned}$$

for some $C_{N,m} \in \mathbb{C}$. One computes that

$$\begin{aligned} 2\partial_{\overline{z}} Q_n(z) &= \sum_{k=0}^{n-1} C_{n,k}(n-k) x^{n-k-1} y^k + \mathbf{i} \sum_{k=1}^{n} C_{n,k} k x^{n-k} y^{k-1} \\ &= \sum_{k=0}^{n-1} C_{n,k}(n-k) x^{n-k-1} y^k + \mathbf{i} \sum_{\tilde{k}=0}^{n-1} C_{n,(\tilde{k}+1)} (\tilde{k}+1) x^{n-\tilde{k}-1} y^{\tilde{k}} \\ &= \sum_{k=0}^{n-1} C_{n,k}(n-k) x^{n-k-1} y^k + \mathbf{i} \sum_{k=0}^{n-1} C_{n,(k+1)} (k+1) x^{n-k-1} y^k \\ &= \sum_{k=0}^{n-1} \big(C_{n,k}(n-k) + \mathbf{i} C_{n,(k+1)}(k+1) \big) x^{n-k-1} y^k. \end{aligned}$$

If P satisfies the Cauchy-Riemann equation (that is, P is analytic), then

$$C_{n,k}(n-k) + \mathbf{i} C_{n,(k+1)}(k+1) = 0 \text{ for all } n = 0,1,\cdots,N \text{ and } k = 0,\cdots,n-1.$$

By using induction, one also can verify that

$$C_{n,k} = \mathbf{i}^k \begin{pmatrix} n \\ k \end{pmatrix} C_{n,0} \quad \text{for all } n = 0,1,\cdots,N \text{ and } k = 0,\cdots,n-1. \tag{2.1.4}$$

Substitute (2.1.4) into P, one reaches

$$P(z) = \sum_{n=0}^{N} C_{n,0} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (\mathbf{i}y)^k = \sum_{n=0}^{N} C_{n,0} (x + \mathbf{i}y)^n = \sum_{n=0}^{N} C_{n,0} z^n. \tag{2.1.5}$$

Combining with Example 2.1.6, we know that a polynomial P enjoys the following property:

$$P \text{ is analytic } \iff P \text{ takes the form (2.1.5)}. \tag{2.1.6}$$

Therefore, if a polynomial takes the form (2.1.5) (or in Example 2.1.6), we called it an *analytic polynomial*.

An application. In one of my research paper [**KLSS22**], we use complex polynomial to construct some explicit examples of domain which is non-scattering with respect to some acoustic wave (which satisfies time-harmonic wave equation).

2.2. Power series

Example 2.1.14 immediately suggests a wider class of direct functions of z, those given by "infinite polynomials" in z:

Definition 2.2.1. A *power series* in z is an infinite series (in the sense of Definition 1.2.10) of the form $\sum_{k=0}^{\infty} C_k z^k$.

We now prove some properties which are similar to the power series on $\mathbb{R}$ (see e.g. [**Rud76**]).

Theorem 2.2.2. *Given a sequence $\{C_k\} \subset \mathbb{C}$.*

(a) *If $\limsup_{k\to\infty} |C_k|^{\frac{1}{k}} = 0$, then $\sum C_k z^k$ converges absolutely for all $z \in \mathbb{C}$. In addition, for each $r > 0$, $\sum C_k z^k$ converges uniformly*[1] *in $z \in B_r$.*
(b) *If $\limsup_{k\to\infty} |C_k|^{\frac{1}{k}} = +\infty$, then $\sum C_k z^k$ converges for $z = 0$ only.*
(c) *If $0 < \limsup_{k\to\infty} |C_k|^{\frac{1}{k}} < +\infty$, then $\sum C_k z^k$ converges absolutely for $|z| < R$ and diverges for $|z| > R$, where*

$$R = \left(\limsup_{k\to\infty} |C_k|^{\frac{1}{k}} \right)^{-1}. \tag{2.2.1}$$

In addition, for each $0 < \epsilon < R$, $\sum C_k z^k$ converges uniformly[2] *in $z \in B_{R-\epsilon}$.*

Remark 2.2.3 (Inconclusive on B_R). For (a) and (b), we simply say the radius of convergence $R = \infty$ and $R = 0$ respectively. The uniform convergence only holds true for $B_{R-\epsilon}$, but not for B_R. If the uniform convergence is on B_R, then the sequence converges on $|z| = R$, however this is not true, see Exercises 2.2.6, 2.2.7 and 2.2.8.

---

[1]The rate of convergence depends on r. We do not know whether the series converges uniformly on the whole $\mathbb{C}$ or not.

[2]The rate of convergence depends on ϵ. We do not know whether the series converges uniformly on the whole B_R or not.

REMARK 2.2.4 (Structure of power series). If $z \in \mathbb{C}$ satisfies $|z| > R$, then by (1.2.7) we have

$$1 < R^{-1}|z| = \limsup_{k\to\infty} |C_k|^{\frac{1}{k}}|z| = \limsup_{k\to\infty} |C_k z^k|^{\frac{1}{k}}.$$

This shows that the sequence $\{C_k z^k\}_{k\in\mathbb{N}}$ does not converge to 0. Otherwise, suppose the contrary that $\{C_k z^k\}_{k\in\mathbb{N}}$ converge to 0, then it must be bounded, says $|C_k z^k| \leq L$ for all k. Hence we see that

$$\limsup_{k\to\infty} |C_k z^k|^{\frac{1}{k}} \leq \limsup_{k\to\infty} L^{\frac{1}{k}} = 1,$$

which is a contradiction. Since $\{C_k z^k\}_{k\in\mathbb{N}}$ does not converge to 0, thus $\sum C_k z^k$ diverges. In view of Theorem 2.2.2, it is make sense to call such constant R is called the *radius of convergence* of the power series $\sum C_k z^k$.

REMARK 2.2.5. By using previous remark, it is important to notice that, if $\sum C_k z^k$ converges at z_0, then it also converges in $B_{|z_0|}$, i.e. the ball with radius $|z_0|$ (not include the boundary, which is inconclusive). Similarly, if $\sum C_k z^k$ diverges at z_0, the it is also diverges in $\mathbb{C} \setminus \overline{B_{|z_0|}}$.

PROOF OF (A). For each $\tilde{r} > 0$, there exists $N > 0$, which depends on $\tilde{r}$, such that

$$|C_k|^{\frac{1}{k}} \leq \frac{1}{2\tilde{r}} \text{ for all } k \geq N \implies |C_k|\tilde{r}^k \leq \frac{1}{2^k} \text{ for all } k \geq N.$$

- For each $z \in \mathbb{C}$, by choosing $\tilde{r} = |z|$, we see that

$$\limsup_{n\to\infty} \sum_{k=n}^{\infty} |C_k z^k| = \limsup_{n\to\infty} \sum_{k\geq n} |C_k| r^k \leq \limsup_{n\to\infty} \sum_{k\geq n} \frac{1}{2^k} = 0,$$

  which concludes that the series converges absolutely at each $z \in \mathbb{C}$.
- On the other hand, for each $r > 0$, one can choose $\tilde{r} = r$ to see that

$$\limsup_{n\to\infty} \sup_{z\in B_r} \left| \sum_{k=n}^{\infty} C_k z^k \right| \leq \limsup_{n\to\infty} \sum_{k\geq n} |C_k| r^k \leq \limsup_{n\to\infty} \sum_{k\geq n} \frac{1}{2^k} = 0,$$

  which concludes that the series converges uniformly in B_r.

□

PROOF OF (B). For any $z \neq 0$, there exists a sequence $\{k_n\} \subset \mathbb{N}$ with $k_n \to +\infty$ such that

$$|C_{k_n}|^{\frac{1}{k_n}} \geq \frac{1}{|z|} \text{ for all } n \implies |C_{k_n} z^{k_n}| = |C_{k_n}||z|^{k_n} \geq 1 \text{ for all } n,$$

which shows that $\sum C_k z^k$ does not converges for all $z \neq 0$ (Note: If $\sum C_k z^k$ converges, then it is necessarily that $C_k z^k \to 0$, which will led a contradiction). □

PROOF OF (C). We first consider the case when $|z| > R$. There exists $\delta > 0$ such that $|z| = R + \delta$, and there exists a sequence $\{k_n\} \subset \mathbb{N}$ with $k_n \to +\infty$ such that

$$|C_{k_n}|^{\frac{1}{k_n}} \geq \frac{1}{R+\delta} \text{ for all } n \implies |C_{k_n} z^{k_n}| = |C_{k_n}||z|^{k_n} \geq 1 \text{ for all } n,$$

so that $\sum C_k z^k$ does not converges. We now fix any $0 < \tilde{r} < R$, and we write $2\delta = R - \tilde{r} > 0$. By using the definition of (2.2.1), one see that there exists $N > 0$, which depends on $\tilde{r}$, such that

$$|C_k|^{\frac{1}{k}} \leq \frac{1}{R-\delta} \text{ for all } k \geq N \implies |C_k|\tilde{r}^k \leq \left(\frac{R-2\delta}{R-\delta}\right)^k \text{ for all } k \geq N.$$

- For each $z \in B_R$, by choosing $\tilde{r} = |z|$, we see that

$$\limsup_{n\to\infty} \sum_{k=n}^{\infty} |C_k z^k| = \limsup_{n\to\infty} \sum_{k=n}^{\infty} |C_k|\tilde{r}^k \leq \limsup_{n\to\infty} \sum_{k=n}^{\infty} \left(\frac{R-2\delta}{R-\delta}\right)^k = 0,$$

  which concludes that the series converges absolutely at each $z \in B_R$.
- On the other hand, for each $0 < \epsilon < R$, we choose $\tilde{r} = R - \epsilon$ to see that

$$\limsup_{n\to\infty} \sup_{z\in B_{R-\epsilon}} \left|\sum_{k=n}^{\infty} C_k z^k\right| \leq \limsup_{n\to\infty} \sum_{k=n}^{\infty} |C_k|\tilde{r}^k \leq \limsup_{n\to\infty} \sum_{k=n}^{\infty} \left(\frac{R-2\delta}{R-\delta}\right)^k = 0,$$

  which concludes that the series converges uniformly in $B_{R-\epsilon}$.

□

When the radius of convergence $R \in (0, \infty)$, there is no guarantee for the convergence or divergence at $z \in \partial B_R$ (however, this is related to Fourier series, see Remark 2.3.7 below). This demonstrates by the following exercises.

Exercise 2.2.6. Show that the radius of convergence of $\sum_{n=1}^{\infty} nz^n$ is $R = 1$, and the series also diverges for $|z| = 1$.

Exercise 2.2.7. Show that the radius of convergence of $\sum_{n=1}^{\infty} n^{-2}z^n$ is $R = 1$, and the series also converges for $|z| = 1$.

Exercise 2.2.8. Show that the radius of convergence of $\sum_{n=1}^{\infty} n^{-1}z^n$ is $R = 1$. In addition, show that the series converges for all $z \in \partial B_1 \setminus \{1\}$ but diverges at $z = 1$.

We now show that power series, like polynomials, are differentiable functions of z (in the sense of Definition 2.1.1).

Theorem 2.2.9. *Suppose that the series $f(z) = \sum_{n=0}^{\infty} C_n z^n$ has the radius of convergence $0 < R \leq +\infty$ given in* (2.2.1) *(see Theorem* (2.2.2)*), then $f'(z)$ exists (in the sense of Definition 2.1.1) and equal to*

$$g(z) := \sum_{n=0}^{\infty} nC_n z^{n-1} \equiv \sum_{n=1}^{\infty} nC_n z^{n-1} \equiv \sum_{m=0}^{\infty} \tilde{C}_m z^m \quad \text{with} \quad \tilde{C}_m := (m+1)C_{m+1} \tag{2.2.2}$$

in B_R, and g also has the radius of convergence R, which is same as f. As an immediate consequence, power series are infinitely differentiable (in the sense of Definition 2.1.1) within their domain of convergence.

Proof. We divide the proof into several steps.

Step 1: Radius of convergence. By using (1.2.6) one sees that

$$\limsup_{m\to\infty} |\tilde{C}_m|^{\frac{1}{m}} = \limsup_{n\to\infty} |nC_n|^{\frac{1}{n-1}} = \lim_{n\to\infty} (n^{\frac{1}{n}})^{\frac{n}{n-1}} \limsup_{n\to\infty} |C_n|^{\frac{1}{n-1}} = \limsup_{n\to\infty} |C_n|^{\frac{1}{n-1}}.$$

There exists a subsequence $\{C_{n_k}\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}|C_n|^{\frac{1}{n-1}} &= \lim_{k\to\infty}|C_{n_k}|^{\frac{1}{n_k-1}} = \lim_{k\to\infty}|C_{n_k}|^{\frac{1}{n_k}\cdot\frac{n_k}{n_k-1}} = \lim_{k\to\infty}|C_{n_k}|^{\frac{1}{n_k}} \\ &\le \lim_{k\to\infty}\sup_{m\ge n_k}|C_m|^{\frac{1}{m}} = \limsup_{n\to\infty}|C_n|^{\frac{1}{n}}.\end{aligned}$$

Conversely, we also can find another subsequence $\{C_{n_\ell}\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}|C_n|^{\frac{1}{n}} &= \lim_{\ell\to\infty}|C_{n_\ell}|^{\frac{1}{n_\ell}} = \lim_{\ell\to\infty}|C_{n_\ell}|^{\frac{1}{n_\ell-1}\cdot\frac{n_\ell-1}{n_\ell}} = \lim_{\ell\to\infty}|C_{n_\ell}|^{\frac{1}{n_\ell-1}} \\ &\le \lim_{\ell\to\infty}\sup_{m\ge n_\ell}|C_m|^{\frac{1}{m-1}} = \limsup_{n\to\infty}|C_n|^{\frac{1}{n-1}}.\end{aligned}$$

Combining the above three equations, we reach

$$\limsup_{m\to\infty}|\tilde{C}_m|^{\frac{1}{m}} = \limsup_{n\to\infty}|C_n|^{\frac{1}{n}},$$

hence we conclude that g also has the radius of convergence R, which is same as f.

Step 2: Show that f' exists and it equal to g. We now further divide our discussions in subcases.

Step 2a: When $R=\infty$. Given any $h\in\mathbb{C}\setminus\{0\}$ with $|h|<1$. The absolute convergence allows us to rearrange the sum, hence

$$\frac{f(z+h)-f(z)}{h}-g(z) = \sum_{n=0}^{\infty}C_n\frac{(z+h)^n-z^n}{h} - \sum_{n=0}^{\infty}nC_nz^{n-1} = \sum_{n=0}^{\infty}C_nb_n,$$

where

$$\begin{aligned}b_n &= \frac{(z+h)^n-z^n}{h}-nz^{n-1} = \frac{1}{h}\left(\sum_{k=0}^{n}\binom{n}{k}h^kz^{n-k}-z^n\right)-nz^{n-1} \quad \text{(binomial theorem)} \\ &= \frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}h^kz^{n-k}-nz^{n-1} = \sum_{k=1}^{n}\binom{n}{k}h^{k-1}z^{n-k}-nz^{n-1} = \sum_{k=2}^{n}\binom{n}{k}h^{k-1}z^{n-k}.\end{aligned}$$

Then

$$\begin{aligned}|b_n| &\le \sum_{k=2}^{n}\binom{n}{k}|h|^{k-1}|z|^{n-k} \le |h|\sum_{k=2}^{n}\binom{n}{k}|z|^{n-k} \\ &\le |h|\sum_{k=0}^{n}\binom{n}{k}|z|^{n-k} = |h|(|z|+1)^n \quad \text{(again binomial theorem)},\end{aligned}$$

and hence

$$\left|\frac{f(z+h)-f(z)}{h}-g(z)\right| \le \sum_{n=0}^{\infty}|C_n||b_n| \le |h|\overbrace{\sum_{n=0}^{\infty}|C_n|(|z|+1)^n}^{<+\infty \text{ because } R=\infty}.$$

Taking $h\to 0$ (in the sense of limit supremum), we conclude f' exists and $f'(z)=g(z)$ for all $z\in\mathbb{C}$.

Step 2b: When $0 < R < \infty$. Given any $|z| < R$, and write $|z| = R - 2\delta$ for some $\delta > 0$. We now let $h \in \mathbb{C} \setminus \{0\}$ with $|h| < \min\{\delta, 1\}$. Then $|z+h| \leq |z| + |h| \leq R - 2\delta + \delta = R - \delta < R$, and as in above, we can write

$$\frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n=0}^{\infty} C_n b_n, \quad b_n = \sum_{k=2}^{\infty} \binom{n}{k} h^{k-1} z^{n-k}.$$

If $z = 0$, then $b_n = h^{n-1}$ and the proof follows easily (left as exercise). We now consider the case when $z \neq 0$. For each $2 \leq k \leq n$ we see that

$$\begin{aligned} \binom{n}{k} &= \frac{n-k+1}{k} \binom{n}{k-1} = \frac{n-k+1}{k} \cdot \frac{n-(k-1)+1}{k-1} \binom{n}{k-2} \\ &\leq \frac{n-2+1}{2} \cdot \frac{n-(2-1)+1}{2-1} \binom{n}{k-2} = \frac{n-1}{2} \cdot n \binom{n}{k-2} \leq n^2 \binom{n}{k-2} \end{aligned}$$

since both $\frac{n-k+1}{k}$ and $\frac{n-(k-1)+1}{k-1}$ are monotone decreasing on k. We now have

$$\begin{aligned} |b_n| &\leq n^2 \sum_{k=2}^{n} \binom{n}{k-2} |h|^{k-1} |z|^{n-k} = \frac{n^2 |h|}{|z|^2} \sum_{k=2}^{n} \binom{n}{k-2} |h|^{k-2} |z|^{n-(k-2)} \\ &= \frac{n^2|h|}{|z|^2} \sum_{j=2}^{n-2} \binom{n}{j} |h|^j |z|^{n-j} \leq \frac{n^2|h|}{|z|^2} \sum_{j=2}^{n} \binom{n}{j} |h|^j |z|^{n-j} \\ &= \frac{n^2|h|}{|z|^2} (|z| + |h|)^n \quad \text{(binomial theorem)} \\ &\leq \frac{n^2|h|}{|z|^2} (R-\delta)^n = \frac{|h|}{|z|^2} \left((R-\delta) n^{\frac{2}{n}} \right)^n, \end{aligned}$$

then we now reach

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \leq \sum_{n=0}^{\infty} |C_n| |b_n| \leq \frac{|h|}{|z|^2} \overbrace{\sum_{n=0}^{\infty} |C_n| \left((R-\delta) n^{\frac{2}{n}} \right)^n}^{< +\infty \text{ since } \sqrt[n]{n} \to 1 \text{ as } n \to \infty}.$$

Taking $h \to 0$ (in the sense of limsup), we conclude f' exists in B_R and $f'(z) = g(z)$ for all $z \in B_R$. ☐

Exercise 2.2.10. Show that if $f(z) = \sum_{n=0}^{\infty} C_n z^n$ has a nonzero radius of convergence, then

$$C_n = \frac{f^{(n)}(0)}{n!} \quad \text{for all } n = 0, 1, 2, \cdots,$$

where $f^{(n)}$ is the n^{th} derivative of f (in the sense of Definition 2.1.1). Show that for each $n = 0, 1, 2, \cdots$ that

$$f^{(n)}(z) = n! C_n + (n+1)! C_{n+1} z + \frac{(n+2)!}{2!} C_{n+2} z^2 + \cdots$$

for all z in the domain of convergence.

THEOREM 2.2.11 (Uniqueness of power series). *Suppose that the power series $f(z) = \sum_{n=0}^{\infty} C_n z^n$ has a nonzero radius of convergence. If there exists a sequence $\{z_k\}$ in the domain of convergence such that*

$$z_k \to 0 \in \mathbb{C}, \quad z_k \neq 0 \text{ and } f(z_k) = 0 \text{ for all } k = 1, 2, 3, \cdots$$

then $f \equiv 0$.

REMARK 2.2.12. If a power series equals to zero at all the points of a set with an accumulation point at the origin, the power series is identically zero in the domain of convergence. As an immediate consequence, if $\sum a_n z^n$ and $\sum b_n z^n$ converge and agree on a set of points with an accumulation point at the origin, then $a_n = b_n$ for all n.

PROOF. We want to show $C_n = 0$ for all $n = 0, 1, 2, \cdots$ by using strong mathematical induction.

- By continuity of f at the origin, we see that
$$C_0 = f(0) = \lim_{z \to \infty} f(z) = \lim_{k \to \infty} f(z_k) = 0.$$
- We now assume the induction hypothesis that $C_j = 0$ for all $j = 0, 1, 2, \cdots, n-1$. The induction hypothesis guarantees that the function
$$g(z) = \frac{f(z)}{z^n} = C_n + C_{n+1} z + C_{n+2} z^2 + \cdots ,$$
is continuous at the origin by defining $g(0) := C_n$. Since
$$0 = \frac{f(z_k)}{z_k^n} = g(z_k) \quad \text{for all } k = 1, 2, 3, \cdots$$
then we conclude our result by taking $k \to \infty$ (so that $z_k \to 0$).

We conclude the theorem by strong mathematical induction. □

2.3. Exponential, sine and cosine functions

We define the exponential function

$$e^z := e^x(\cos\theta + \mathbf{i}\sin\theta) \quad \text{for all } z = x + \mathbf{i}\theta \in \mathbb{C}. \tag{2.3.1}$$

It is easy to see that $|e^z| = e^x$ and $e^z \neq 0$ for all $z = x + \mathbf{i}y \in \mathbb{C}$.

EXERCISE 2.3.1. Prove that $e^{z_1+z_2} = e^{z_1} e^{z_2}$ for all $z_1, z_2 \in \mathbb{C}$.

Euler's formula is just a special case of (2.3.1):

$$e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta \quad \text{for all } \theta \in \mathbb{R}. \tag{2.3.2}$$

EXERCISE 2.3.2 (Euler, De Moivre). For each $n \in \mathbb{N}$, show that $(\cos\theta + \mathbf{i}\sin\theta)^n = \cos(n\theta) + \mathbf{i}\sin(n\theta)$ for all $\theta \in \mathbb{R}$.

It is useful to see that

$$z = |z| e^{\mathbf{i}\theta} \quad \text{for all } z \in \mathbb{C} \tag{2.3.3}$$

for some $\theta \in [0, 2\pi)$, which is just simply the polar coordinate in $\mathbb{R}^2$.

EXERCISE 2.3.3. Show that e^z is entire (Definition 2.1.1) by verifying the Cauchy-Riemann equation.

EXERCISE 2.3.4. Prove that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

with radius of convergence $R = +\infty$.

By using Exercise 2.3.3 and Remark 2.1.12, one can easily see that

$$\partial_z e^z = \partial_x (e^x(\cos y + \mathbf{i} \sin y)) = e^x(\cos y + \mathbf{i} \sin y) = e^z \quad \text{for all } z = x + \mathbf{i}y \in \mathbb{C}.$$

From (2.3.2), we see that

$$\begin{aligned} \sin\theta &:= \frac{1}{2\mathbf{i}}(e^{\mathbf{i}\theta} - e^{-\mathbf{i}\theta}) \quad \text{for all } \theta \in \mathbb{R}, \\ \cos\theta &:= \frac{1}{2}(e^{\mathbf{i}\theta} + e^{-\mathbf{i}\theta}) \quad \text{for all } \theta \in \mathbb{R}. \end{aligned}$$

Therefore it is natural to define the entire functions

$$\sin z := \frac{1}{2\mathbf{i}}(e^{\mathbf{i}z} - e^{-\mathbf{i}z}) \quad \text{for all } z \in \mathbb{C}, \tag{2.3.4a}$$

$$\cos z := \frac{1}{2}(e^{\mathbf{i}z} + e^{-\mathbf{i}z}) \quad \text{for all } z \in \mathbb{C}. \tag{2.3.4b}$$

We remind the readers that $\cos z$ and $\sin z$ are not bounded in modulus by 1, since

$$\begin{aligned} \sin(\mathbf{i}\theta) &= \frac{1}{2\mathbf{i}}(e^{-\theta} - e^{\theta}) \quad \text{for all } \theta \in \mathbb{R}, \\ \cos(\mathbf{i}\theta) &= \frac{1}{2}(e^{-\theta} + e^{\theta}) \quad \text{for all } \theta \in \mathbb{R}. \end{aligned}$$

EXERCISE 2.3.5. Show that $\sin z$ and $\cos z$ are entire (Definition 2.1.1) by verifying the Cauchy-Riemann equation. Verify the identities

$$\sin 2z = 2 \sin z \cos z, \quad \sin^2 z + \cos^2 z = 1, \quad (\sin z)' = \cos z.$$

Compute $(\cos z)'$.

EXERCISE 2.3.6. Show that $\sin z$ is entire by proving

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

with radius of convergence $R = +\infty$. Show that $\cos z$ is entire by finding its power series representation and compute its radius of convergence.

Finally, we end this chapter by the following remark.

REMARK 2.3.7. Let $\sum_n c_n z^n$ be a power series with radius of convergence of $0 < R < \infty$, as described in Theorem 2.2.2. We do not know whether the power series converges on $z \in \partial B_R$ or not, see Exercise 2.2.6, Exercise 2.2.7 and Exercise 2.2.8. For each $z \in \partial B_R$, we can write $z = Re^{\mathbf{i}\theta}$, and plut this form into the power series to obtain

$$\sum_n \tilde{c}_n e^{\mathbf{i}n\theta} \quad \text{with} \quad \tilde{c}_n = c_n R^n. \tag{2.3.5}$$

The series (2.3.5) is indeed a special case of Fourier series of period 2π, see e.g. my previous lecture note [**Kow22**] for further details.

CHAPTER 3

Integration

In previous chapter, we are focusing in (complex) differentiability of complex-valued functions. We now discuss its counterpart: the integral.

3.1. The fundamental theorem of line integral

Before we consider the function with complex domain, we first deal with the functions defined on interval in $\mathbb{R}$.

DEFINITION 3.1.1. Let $\phi \equiv \mathfrak{Re}\,\phi + \mathbf{i}\mathfrak{Im}\,\phi : [a,b] \subset \mathbb{R} \to \mathbb{C}$, which is continuous on $[a,b]$, that is $\mathfrak{Re}\,\phi, \mathfrak{Im}\,\phi \in C([a,b])$. The integral of ϕ is defined by

$$\int_a^b \phi(t)\,\mathrm{d}t := \int_a^b \mathfrak{Re}\,\phi(t)\,\mathrm{d}t + \mathbf{i}\int_a^b \mathfrak{Im}\,\phi(t)\,\mathrm{d}t,$$

where $\int_a^b \mathfrak{Re}\,\phi(t)\,\mathrm{d}t$ and $\int_a^b \mathfrak{Im}\,\phi(t)\,\mathrm{d}t$ are just the usual Riemann integral.

DEFINITION 3.1.2. Let $\mathcal{C} = \left[\ z(t) = x(t) + \mathbf{i}y(t) \;\middle|\; a \le t \le b\ \right]$ be an (oriented) continuous curve in $\mathbb{C}$. If x and y are both differentiable at some $t \in (a,b)$, then we set

$$\dot{z}(t) := x'(t) + \mathbf{i}y'(t) \quad \text{for such } t.$$

We call the curve $\mathcal{C}$ is *piecewise-C^1* if both $x, y \in C([a,b])$ and $x, y \in C^1$ on each subinterval $\mathrm{int}\,[a,x_1]$, $\mathrm{int}\,[x_1,x_2]$, ..., $\mathrm{int}\,[x_{n-1},b]$ of some partition of $[a,b]$. If in addition that $\dot{z}(t) \neq 0$ for all but finitely many $t \in (a,b)$ (i.e. there are at most finitely many t_0 such that $x'(t_0) = y'(t_0) = 0$), then we called it a *parametrizable continuous piecewise-C^1 curve.*

REMARK 3.1.3. Usually we refer an oriented curve $\mathcal{C}$ smooth when both $x, y \in C^\infty(a,b)$. Therefore here we will not follow the terminology in [**BN10**].

Finally, we define the important concept of a line integral. This concept also introduced in the vector calculus, see e.g. [**GM12**].

DEFINITION 3.1.4 (Line integral). Let $\mathcal{C} = \left[\ z(t) \;\middle|\; a \le t \le b\ \right]$ be a parametrizable continuous piecewise-C^1 curve and suppose the complex-valued function f is continuous on $\mathcal{C}$ (up to endpoints). The (*line*) *integral* of f along $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} f \equiv \int_{\mathcal{C}} f(z)\,\mathrm{d}z := \int_a^b f(z)\big|_{z=z(t)}\,\dot{z}(t)\,\mathrm{d}t = \int_a^b f(z(t))\dot{z}(t)\,\mathrm{d}t,$$

where the integrand (i.e. the quantity being integrated) is the complex multiplication of $f(z(t))$ and $\dot{z}(t)$.

It is clear that the integral depends on the curve $\mathcal{C}$, more precisely, the integral depends on parametrization z (hence depends on its orientation). Therefore we denote $\mathcal{C} = \left[\ z(t) \;\middle|\; a \le t \le b\ \right]$ rather than $\left\{\ z(t) \;\middle|\; a \le t \le b\ \right\}$ to emphasize the orientation of the curve. However, it is possible to perturb the integral curve without changing the values of the line integral $\int_{\mathcal{C}} f$.

LEMMA 3.1.5. *Let* $\mathcal{C}_1 = \left[\ z(t) \ \middle|\ a \leq t \leq b\ \right]$ *and* $\mathcal{C}_2 = \left[\ w(t) \ \middle|\ c \leq t \leq d\ \right]$ *be two parametrizable continuous piecewise-*C^1 *curves in* $\mathbb{C}$. *If there exists an injective* C^1 *mapping* $\lambda : [c,d] \to [a,b]$ *such that*

$$\lambda(c) = a, \quad \lambda(d) = b, \quad w(t) = z(\lambda(t)) \text{ for all } t \in [c,d], \tag{3.1.1}$$

then $\int_{\mathcal{C}_1} f = \int_{\mathcal{C}_2} f$.

EXERCISE 3.1.6. Prove Lemma 3.1.5.

EXERCISE 3.1.7. Let $\mathcal{C}_1 = \left[\ z(t) \ \middle|\ a \leq t \leq b\ \right]$ and $\mathcal{C}_2 = \left[\ w(t) \ \middle|\ c \leq t \leq d\ \right]$ be two parametrizable continuous piecewise-C^1 curves in $\mathbb{C}$. We define the relation $\sim$ by

$$\mathcal{C}_1 \sim \mathcal{C}_2 \iff \text{there exists } \lambda \in C^1([c,d]) \text{ satisfies (3.1.1)}. \tag{3.1.2}$$

Show that $\sim$ is an equivalence relation, i.e. show that:

(1) **Reflexivity.** $\mathcal{C} \sim \mathcal{C}$ for any parametrizable continuous piecewise-C^1 curve $\mathcal{C}$ in $\mathbb{C}$.
(2) **Symmetry.** $\mathcal{C}_1 \sim \mathcal{C}_2 \iff \mathcal{C}_2 \sim \mathcal{C}_1$ for all parametrizable continuous piecewise-C^1 curves $\mathcal{C}_1, \mathcal{C}_2$ in $\mathbb{C}$.
(3) **Transitivity.** Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be parametrizable continuous piecewise-C^1 curves in $\mathbb{C}$. If $\mathcal{C}_1 \sim \mathcal{C}_2$ and $\mathcal{C}_2 \sim \mathcal{C}_3$, then $\mathcal{C}_1 \sim \mathcal{C}_3$.

Therefore, we can rephrase Lemma 3.1.5 as: If $\mathcal{C}_1$ and $\mathcal{C}_2$ are parametrizable continuous piecewise-C^1 curves in $\mathbb{C}$ which are equivalent in the sense of (3.1.2), then $\int_{\mathcal{C}_1} f = \int_{\mathcal{C}_2} f$.

LEMMA 3.1.8. *Let* $\mathcal{C} = \left[\ z(t) \ \middle|\ a \leq t \leq b\ \right]$ *be a parametrizable continuous piecewise-*C^1 *curve in* $\mathbb{C}$. *If we define*

$$\mathcal{C}^{\mathrm{rev}} := \left[\ z(b+a-t) \ \middle|\ a \leq t \leq b\ \right],$$

then $\int_{\mathcal{C}^{\mathrm{rev}}} f = -\int_{\mathcal{C}} f$.

One should notice that, $\mathcal{C}$ and $\mathcal{C}^{\mathrm{rev}}$ are identical as sets, but reverse oriented.

EXERCISE 3.1.9. Prove Lemma 3.1.8.

The following lemma exhibit a basic property of line integral.

LEMMA 3.1.10. *Let* $\mathcal{C}$ *be a parametrizable continuous piecewise-*C^1 *curve, then the mapping* $f \mapsto \int_{\mathcal{C}} f$ *is* $\mathbb{C}$*-linear, that is,*

(1) $\int_{\mathcal{C}} (f+g) = \int_{\mathcal{C}} f + \int_{\mathcal{C}} g$ *for all* $f, g \in C(\mathcal{C})$,
(2) $\int_{\mathcal{C}} \alpha f = \alpha \int_{\mathcal{C}} f$ *for all* $f \in C(\mathcal{C})$ *and* $\alpha \in \mathbb{C}$.

Here $C(\mathcal{C})$ *denotes the collection of continuous functions on* $\mathcal{C}$ *(up to endpoints).*

EXERCISE 3.1.11. Prove Lemma 3.1.10.

LEMMA 3.1.12. *If the complex-valued function* $G \in C([a,b])$, *then*

$$\left| \int_a^b G(t) \, \mathrm{d}t \right| \leq \int_a^b |G(t)| \, \mathrm{d}t.$$

The LHS of the above integral is defined in the sense of Definition 3.1.1, while the RHS is the usual Riemann integral.

PROOF. We first write $\int_a^b G(t)\,\mathrm{d}t$ in terms of polar coordinate, that is,

$$\int_a^b G(t)\,\mathrm{d}t = \left|\int_a^b G(t)\,\mathrm{d}t\right| e^{i\theta}$$

for some $\theta \in [0,2\pi)$. By linearity of $\int_{\mathcal{C}}$, we reach

$$\left|\int_a^b G(t)\,\mathrm{d}t\right| = \int_a^b e^{-i\theta}G(t)\,\mathrm{d}t \overset{\text{def}}{\equiv} \int_a^b \mathfrak{Re}\left(e^{-i\theta}G(t)\right)\,\mathrm{d}t + \mathbf{i}\int_a^b \mathfrak{Im}\left(e^{-i\theta}G(t)\right)\,\mathrm{d}t.$$

Taking real part of the above equation, we reach

$$\left|\int_a^b G(t)\,\mathrm{d}t\right| = \int_a^b \mathfrak{Re}\left(e^{-i\theta}G(t)\right)\,\mathrm{d}t.$$

Since $\left|\mathfrak{Re}\left(e^{-i\theta}G(t)\right)\right| \leq |e^{-i\theta}G(t)| = |G(t)|$, by the monotonicity of the usual Riemann integral, we reach

$$\left|\int_a^b G(t)\,\mathrm{d}t\right| = \int_a^b \mathfrak{Re}\left(e^{-i\theta}G(t)\right)\,\mathrm{d}t \leq \int_a^b |G(t)|\,\mathrm{d}t,$$

which is our desired result. □

LEMMA 3.1.13. *Let $\mathcal{C}$ be a parametrizable continuous piecewise-C^1 curve with length $\mathscr{H}^1(\mathcal{C})$, then*

$$\left|\int_{\mathcal{C}} f\right| \leq \|f\|_{L^\infty(\mathcal{C})}\mathscr{H}^1(\mathcal{C}) \quad \text{for all } f \in C(\mathcal{C}).$$

REMARK 3.1.14. This implies that, although $\int_{\mathcal{C}} f$ depends on the parametrization of $\mathcal{C}$, it is possible to find an upper bound which is independent of parametrization. In particular, the length of the parametrizable continuous piecewise-C^1 curve is exactly identical to its 1-dimensional Hausdorff measure [**BBI01**, Theorem 2.6.2].

PROOF OF LEMMA 3.1.13. Write $\mathcal{C} = \left[\ z(t) \mid a \leq t \leq b\ \right]$, and recall that

$$\mathscr{H}^1(\mathcal{C}) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t = \int_a^b |\dot{z}(t)|\,\mathrm{d}t.$$

By Lemma 3.1.12 we see that

$$\left|\int_{\mathcal{C}} f\right| = \left|\int_a^b f(z(t))\dot{z}(t)\,\mathrm{d}t\right| \leq \int_a^b |f(z(t))||\dot{z}(t)|\,\mathrm{d}t \leq \|f\|_{L^\infty(\mathcal{C})}\int_a^b |\dot{z}(t)|\,\mathrm{d}t.$$

We combine the above two equations and conclude the lemma. □

LEMMA 3.1.15. *Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ uniformly on the parametrizable continuous piecewise-C^1 curve $\mathcal{C}$. Then*

$$\int_{\mathcal{C}} f = \lim_{n\to\infty}\int_{\mathcal{C}} f_n.$$

PROOF. By (1.2.7), linearity of $\int_{\mathcal{C}}$ and Lemma 3.1.13, one easily sees that

$$\begin{aligned}\limsup_{n\to\infty}\left|\int_{\mathcal{C}} f(z)\,\mathrm{d}z - \int_{\mathcal{C}} f_n(z)\,\mathrm{d}z\right| &= \limsup_{n\to\infty}\left|\int_{\mathcal{C}} (f(z) - f_n(z))\,\mathrm{d}z\right| \\ &\leq \mathscr{H}^1(\mathcal{C})\limsup_{n\to\infty}\|f - f_n\|_{L^\infty(\mathcal{C})} = 0,\end{aligned}$$

which conclude our lemma. □

We now prove one of the main result of this section, which also can be regard as a generalization of fundamental theorem of calculus (integral operator as an inverse operator of differentiation operator).

THEOREM 3.1.16 (Fundamental theorem of line integral). *Let* $\mathcal{C} = \left[\ z(t) \mid a \leq t \leq b\ \right]$ *be a parametrizable continuous piecewise-*C^1 *curve. If* $f \in C^1(\mathcal{C})$ *is (complex) differentiable on* $\mathcal{C}$*, then*

$$\int_{\mathcal{C}} f' = f(z(b)) - f(z(a)).$$

REMARK 3.1.17. The C^1 assumption on f is to ensure that $f' \in C(\mathcal{C})$ so that $\int_{\mathcal{C}} f'$ is well-defined according to Definition 3.1.4.

PROOF OF THEOREM 3.1.16. By assumptions, we have $\dot{z}(t) \neq 0$ for all but finitely many $a < t < b$. For such t, we can find $\delta_t > 0$ so that $z(t+h) - z(t) \neq 0$ and $a < t+h < b$ for all $|h| < \delta_t$. We see that see that

$$\frac{f(z(t+h)) - f(z(t))}{h} = \frac{f(z(t+h)) - f(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h} \quad \text{for all } 0 < |h| < \delta_t,$$

which gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(f(z(t))) &= \lim_{\mathbb{R} \ni h \to 0} \frac{f(z(t+h)) - f(z(t))}{h} \\
&= \lim_{\mathbb{R} \ni h \to 0} \frac{f(z(t+h)) - f(z(t))}{z(t+h) - z(t)} \cdot \lim_{\mathbb{R} \ni h \to 0} \frac{z(t+h) - z(t)}{h} \\
&= \lim_{\mathbb{C} \ni w \to z(t)} \frac{f(w) - f(z(t))}{w - z(t)} \cdot \lim_{\mathbb{R} \ni h \to 0} \frac{z(t+h) - z(t)}{h} \\
&= f'(z)\big|_{z=z(t)} \, \dot{z}(t) \quad \text{(complex multiplication)}.
\end{aligned}$$

Hence by the definition of line integral, one sees that

$$\int_{\mathcal{C}} f'(z) \,\mathrm{d}z \overset{\text{def}}{\equiv} \int_a^b f'(z)\big|_{z=z(t)} \, \dot{z}(t) \,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f(z(t))) \,\mathrm{d}t = f(z(b)) - f(z(a)),$$

where the last equality is just simply the fundamental theorem of calculus [**Rud76**, Theorem 6.21]. □

3.2. Cauchy closed curve theorem in rectangle

We begin our discussions by the following definition.

DEFINITION 3.2.1. A parametrizable continuous piecewise-C^1 curve $\mathcal{C} = \left[\ z(t) \mid a \leq t \leq b\ \right]$ is *closed* if $z(a) = z(b)$. If, in addition, $z(t_1) \neq z(t_2)$ for all $t_1 < t_2$ with $(t_1, t_2) \neq (a, b)$, then we call such closed curve *simple*.

REMARK 3.2.2. The curve with shape "∞" is closed but not simple.

Here not to be confused with the terminology "topological closed". For example, a straight line with finite length is topological closed, but not closed in the sense of Definition 3.2.1. For later convenience, we again clarify the following notion (despite we already introduced before):

DEFINITION 3.2.3. Let $\mathcal{K}$ be a topological closed set in $\mathbb{C}$. We say that f is analytic *near* $\mathcal{K}$ if there exists an open neighborhood Ω of $\mathcal{K}$ (i.e. an open set Ω such that $\mathcal{K} \subset \Omega$) such that f is analytic in Ω. If $\mathcal{K} = \{z\}$ is a one point set, then we say that f is analytic *near* z. In particular, one sees that f is analytic near z if and only if there exists $\epsilon > 0$ such that f is analytic in the ball $B_\epsilon(z)$.

The main theme of this section is to prove the following result, which somehow can be view as a generalization of Exercise 3.1.7, see also [**GM12**, Theorems 6.6.2 and 6.6.3] for analogous result on vector fields on $\mathbb{R}^n$.

THEOREM 3.2.4 (Cauchy closed curve theorem in rectangle). *Let $\mathcal{C}$ be a parametrizable continuous piecewise-C^1 closed curve. If f is analytic near a topological closed rectangle $\mathcal{R}$ such that $\mathcal{C} \subset \mathcal{R}$, then $\int_{\mathcal{C}} f = 0$.*

REMARK 3.2.5. The above theorem holds true for any parametrization of the curve $\mathcal{C}$. The main point here is f has no singularity in the area enclosed by the curve $\mathcal{C}$. If f has some singularity inside it, then the above theorem does not hold. We will discuss such cases later in Chapter 5. We also also prove a fairly general version of the Cauchy closed curve theorem later in Section 3.3.

LEMMA 3.2.6. *Let $\mathcal{C}$ be a parametrizable continuous piecewise-C^1 closed curve. If $f(z) = \alpha + \beta z$ for some $\alpha, \beta \in \mathbb{C}$ (that is, a linear function), then $\int_{\mathcal{C}} f = 0$.*

PROOF. If we define $F(z) := \alpha z + \frac{1}{2}\beta z^2$, by using Exercise 2.1.6, one has $F' = f$. By writing $\mathcal{C} = \left[\ z(t) \ \middle|\ a \leq t \leq b\ \right]$ and using the fundamental theorem of line integral (Theorem 3.1.16), one sees that

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}} F' = F(z(b)) - F(z(a)) = 0,$$

which immediately conclude our lemma. □

EXERCISE 3.2.7. Let $\{\mathcal{K}^{(k)}\}$ be a sequence of compact sets in $\mathbb{C} \cong \mathbb{R}^2$ such that $\mathcal{K}^{(1)} \supset \mathcal{K}^{(2)} \supset \mathcal{K}^{(3)} \supset \cdots$. Show that $\bigcap_{k\in\mathbb{N}} \mathcal{K}^{(k)} \neq \emptyset$. [Hint: consider the complement of $\mathcal{K}^{(k)}$.]

We now prove the following technical lemma.

LEMMA 3.2.8 (Rectangle lemma). *Let Γ be the boundary of a topological closed rectangle $\mathcal{R}$. If f is analytic near $\mathcal{R}$, then $\int_\Gamma f = 0$.*

PROOF. Without loss of generality, we may choose a parametrization of Γ in counterclockwise orientation, since the reverse orientation will gives a minus sign (Lemma 3.1.8), which does not affect our lemma at all.

We split the topological closed rectangle $\mathcal{R}$ into 4 congruent subrectangles, by bisecting each of the sides. We let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ denote the boundaries (counterclockwise orientation) of the four topological closed subrectangles (also in counterclockwise order) as in the following figure:

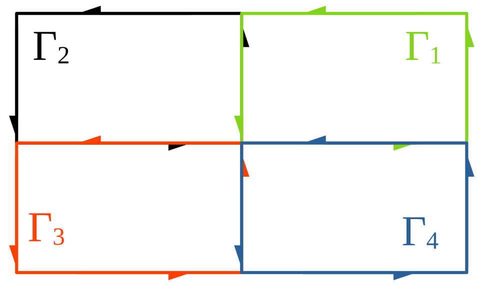


FIGURE 3.2.1. Splitting a rectangle into 4 congruent subrectangles

Since the integrals along the interior lines appear in the opposite directions and thus cancel (Lemma 3.1.8), hence we see that

$$\int_{\Gamma} f = \sum_{i=1}^{4} \int_{\Gamma_i} f.$$

From this, one sees that

$$\left| \int_{\Gamma_i} f \right| \geq \frac{1}{4} \left| \int_{\Gamma} f \right| \quad \text{for } \textbf{some } i = 1,2,3,4.$$

We denote $\Gamma^{(0)} = \Gamma$ and $\Gamma^{(1)} = \Gamma_i$ for i which satisfies the above inequality. Let $\mathcal{R}^{(1)}$ the topological closed rectangle enclosed by the closed curve $\Gamma^{(1)}$. We now show (by using mathematical induction) that one can obtain a sequence of topological closed rectangles

$$\mathcal{R}^{(1)} \supset \mathcal{R}^{(2)} \supset \mathcal{R}^{(3)} \supset \cdots \quad \text{with} \quad \left| \int_{\Gamma^{(k)}} f \right| \geq \frac{1}{4^k} \left| \int_{\Gamma} f \right|, \tag{3.2.1}$$

where $\Gamma^{(k)}$ is the boundary of the $\mathcal{R}^{(k)}$.

We already show (3.2.1) when $k = 1$. We now assume the induction hypothesis that (3.2.1) holds for $k = \ell$. We now splitting the topological closed rectangle $\mathcal{R}^{(\ell)}$ into $\mathcal{R}_1^{(\ell)}, \mathcal{R}_2^{(\ell)}, \mathcal{R}_3^{(\ell)}, \mathcal{R}_4^{(\ell)}$ as in Figure 3.2.1, where $\Gamma_i^{(\ell)}$ are boundary of $\mathcal{R}_i^{(\ell)}$. We again see that

$$\int_{\Gamma^{(\ell)}} f = \sum_{i=1}^{4} \int_{\Gamma_i^{(\ell)}} f.$$

From this, one sees that

$$\left| \int_{\Gamma_i^{(\ell)}} f \right| \geq \frac{1}{4} \left| \int_{\Gamma^{(\ell)}} f \right| \geq \frac{1}{4^{\ell+1}} \left| \int_{\Gamma} f \right| \quad \text{for } \textbf{some } i = 1,2,3,4.$$

We now denote choose $\Gamma^{(\ell+1)} := \Gamma_i^{(k)}$ for i satisfies the above inequality and $\mathcal{R}^{(\ell+1)}$ be the topological closed rectangle enclosed by the closed curve $\Gamma^{(\ell+1)}$. We now complete the proof of (3.2.1) by induction.

By using Exercise 3.2.7, we know that $\bigcap_{k\in\mathbb{N}} \mathcal{R}^{(k)} \neq \emptyset$. We now fix one $z_0 \in \bigcap_{k\in\mathbb{N}} \mathcal{R}^{(k)}$. One sees that

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \iff \lim_{z\to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0} = 0.$$

For later convenience, we denote

$$\mathsf{o}_z := \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0}$$

so that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \mathsf{o}_z \cdot (z - z_0), \quad \lim_{z \to z_0} \mathsf{o}_z = 0.$$

By using Lemma 3.2.6, we see that

$$\int_{\Gamma^{(n)}} f = \int_{\Gamma^{(n)}} \mathsf{o}_z \cdot (z - z_0)\, \mathrm{d}z.$$

Let s be the largest side of the original boundary Γ (so that $\mathscr{H}^1(\Gamma) \leq 4s$ and $|z - z_0| \leq \sqrt{2}s$), then

$$\mathscr{H}^1(\Gamma^{(n)}) = \frac{1}{2^n}\mathscr{H}^1(\Gamma) \leq \frac{4s}{2^n}, \quad \sup_{z \in \Gamma^{(n)}} |z - z_0| \leq \frac{\sqrt{2}s}{2^n}.$$

By the definition of o_z, given any $\epsilon > 0$, there exists N such that

$$|\mathsf{o}_z| \leq \epsilon \quad \text{for all } |z - z_0| \leq \frac{\sqrt{2}s}{2^N},$$

which shows that

$$\sup_{z \in \Gamma^{(n)}} |\mathsf{o}_z| \leq \epsilon \quad \text{for all } n \geq N.$$

By using Lemma 3.1.13 and (3.2.1), by fixing any $n \geq N$, we see that

$$\left| \int_\Gamma f \right| \leq 4^n \left| \int_{\Gamma^{(n)}} f \right| = 4^n \left| \int_{\Gamma^{(n)}} \mathsf{o}_z \cdot (z - z_0)\, \mathrm{d}z \right| \leq \epsilon 4\sqrt{2}s^2.$$

We see that the first and last terms of the above are independent of N. By arbitrariness of ϵ, we conclude our lemma. □

We now prove an important theorem, which is analogue to the fundamental theorem of calculus (antiderivative).

THEOREM 3.2.9 (Fundamental theorem of antiderivative in rectangle). *If f is analytic near a topological closed rectangle $\mathcal{R}$, then there exists a function F which is analytic and $F' = f$ near $\mathcal{R}$. Such analytic function F is called the (complex)* antiderivative *of f. Combining this with the fundamental theorem of line integral (Theorem 3.1.16), we have*

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}} F' = F(z(b)) - F(z(a)) \tag{3.2.2}$$

for any parametrizable continuous piecewise-C^1 curve $\mathcal{C} = \left[\ z(t) \mid a \leq t \leq b\ \right] \subset \mathcal{R}$.

REMARK 3.2.10. We will later show a fairly general version of the above theorem in Theorem 3.3.10 later.

PROOF OF THEOREM 3.2.9. Without loss of generality, we may assume $0 \in \mathcal{R}$. We define

$$F(z) := \int_0^z f(\zeta)\, \mathrm{d}\zeta \equiv \int_{\mathcal{C}_1} f(\zeta)\, \mathrm{d}\zeta, \tag{3.2.3}$$

where $\mathcal{C}_1$ denotes the oriented curve consists of the straight lines from 0 to $\mathfrak{Re}\, z$ and then from $\mathfrak{Re}\, z$ to z. For each $h \in \mathbb{C}$, we also denote

$$\int_z^{z+h} f(\zeta)\,\mathrm{d}\zeta \equiv \int_{\mathcal{C}_2} f(\zeta)\,\mathrm{d}\zeta,$$

where $\mathcal{C}_2$ denotes the oriented curve consists of the straight lines from z to $z+\mathfrak{Re}\, h$ and then from $z+\mathfrak{Re}\, h$ to $z+h$. By the definition (3.2.3), we have

$$F(z+h) = \int_0^{z+h} f(\zeta)\,\mathrm{d}\zeta = \int_{\mathcal{C}_3} f(\zeta)\,\mathrm{d}\zeta,$$

where $\mathcal{C}_3$ denotes the oriented curve consists of the straight lines from 0 to $\mathfrak{Re}\,(z+h)$ and then from $\mathfrak{Re}\,(z+h)$ to $z+h$. In particular, one sees that

$$F(z) + \int_z^{z+h} f(\zeta)\,\mathrm{d}\zeta = F(z+h),$$

see the following figure:

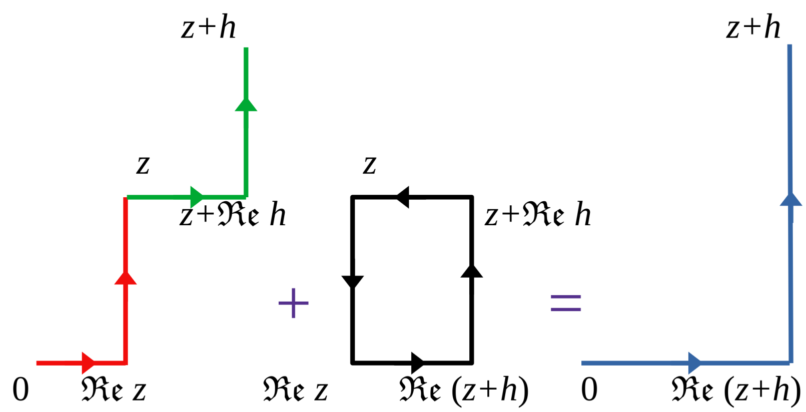


FIGURE 3.2.2. The sketch of the curves $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$

Since

$$F(z+h) - F(z) = \int_z^{z+h} f(\zeta)\,\mathrm{d}\zeta$$

and

$$\frac{1}{h}\int_z^{z+h} 1\,\mathrm{d}z = \frac{1}{h}\overbrace{((z+\mathfrak{Re}\, h) - z)}^{\text{from } z \text{ to } z+\mathfrak{Re}\, h} + \frac{1}{h}\overbrace{((z+h) - (z+\mathfrak{Re}\, h))}^{\text{from } z+\mathfrak{Re}\, h \text{ to } z+h} = 1,$$

then

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} (f(\zeta) - f(z))\,\mathrm{d}\zeta.$$

Since $\mathscr{H}^1(\mathcal{C}_2) = |\mathfrak{Re}\, h| + |\mathfrak{Im}\, h| \leq 2|h|$, finally by Lemma 3.1.13 and the (uniform) continuity of f, we have

$$\begin{aligned}\limsup_{\mathbb{C}\ni h\to 0}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &\leq \limsup_{\mathbb{C}\ni h\to 0}\frac{1}{|h|}\|f - f(z)\|_{L^\infty(\mathcal{C}_2)}\mathscr{H}^1(\mathcal{C}_2)\\ &\leq 2\limsup_{\mathbb{C}\ni h\to 0}\|f - f(z)\|_{L^\infty(\mathcal{C}_2)} = 0\end{aligned}$$

which conclude our theorem. ☐

With this fundamental theorem at hand, we finally now ready to prove the main result of this section, that is, the Cauchy closed curve theorem in rectangle.

PROOF OF THEOREM 3.2.4. Write $\mathcal{C} = \left[\ z(t) \ \middle| \ a \leq t \leq b \ \right] \subset \mathcal{R}$ with $z(a) = z(b)$. Since f is analytic near $\mathcal{R}$, by the fundamental theorem of antiderivative in rectangle (Theorem 3.2.9), there exists a function F which is analytic and $F' = f$ near $\mathcal{R}$ such that

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}} F' = F(z(b)) - F(z(a)) = 0,$$

which immediately conclude the theorem. ☐

3.3. Cauchy closed curve theorem in simply connected open sets

In this section will prove a version of Cauchy closed curve theorem, which generalized Theorem 3.2.4. The main theme of this section is to remove the analyticity assumption on rectangles. Let A and B are sets, then we denote the *distance* between them by

$$\mathrm{dist}\,(A,B) = \inf_{a \in A, b \in B} |a - b|.$$

If A is a one point set $\{z_0\}$, we simply denote $\mathrm{dist}\,(z_0, B)$.

DEFINITION 3.3.1. Let Ω be an open set. If Ω is connected and its complement is "connected to ∞ by a continuous curve within ϵ-neighborhood of $\mathbb{C} \setminus \Omega$" in the following sense: if for any $z_0 \notin \Omega$ and $\epsilon > 0$, there is a continuous curve $\gamma = [\gamma(t) : 0 \leq t < \infty]$ such that

$$\mathrm{dist}\,(\gamma(t), \mathbb{C} \setminus \Omega) < \epsilon \text{ for all } t \geq 0, \quad \gamma(0) = z_0, \quad \lim_{t \to \infty} |\gamma(t)| = \infty,$$

then we call such set Ω is *simply connected open set* in $\mathbb{C}$.

EXAMPLE 3.3.2. The annulus $\left\{ \ z \in \mathbb{C} \ \middle| \ 1 < |z| < 3 \ \right\}$ is not simply connected, because its complement cannot be "connected to ∞ by a continuous curve within ϵ-neighborhood of $\mathbb{C} \setminus \Omega$".

EXAMPLE 3.3.3. The infinite strip $S = \left\{ \ z \in \mathbb{C} \ \middle| \ -1 < \mathfrak{Im}\, z < 1 \ \right\}$ is connected. Note that in this case, $\mathbb{C} \setminus S$ is not connected.

EXERCISE 3.3.4. A set S is called *star-like* if there exists a point $\alpha \in S$ such that the line segment connecting α and z is contained in S for all $z \in S$. Show that a star-like region is simply connected.

We now exhibit an example to demonstrate the generality of Definition 3.3.1.

EXAMPLE 3.3.5. The complement of the connected domain

$$\left\{ \ x + \mathbf{i}y \in \mathbb{C} \ \middle| \ 0 < x \leq 1, y = \sin \tfrac{1}{x} \ \right\} \cup \left\{ \ \mathbf{i}y \in \mathbb{C} \ \middle| \ -1 < y < \infty \ \right\}$$

is simply connected.

DEFINITION 3.3.6. Let Γ be a polygonal path (Definition 1.2.16) consists of horizontal lines and vertical lines, i.e. either parallel to real axis or parallel to v the imaginary axis. The y_0*-level* is the set

$$\Gamma_{y_0} := \left\{ \ x + \mathbf{i}y_0 \ \middle| \ x \in \mathbb{R} \ \right\} \cap \Gamma.$$

If $\{\Gamma_{y_1}, \cdots, \Gamma_{y_n}\}$, for some $y_1 > y_2 > \cdots > y_n$, are all levels of Γ, then we say that $n \in \mathbb{N}$ is the *number of levels*. We also say Γ_{y_1} the *top level* of Γ. We also say that $\Gamma_{y_{j+1}}$ is the *next level* of Γ_{y_j}.

EXERCISE 3.3.7. Let K be a compact set in $\mathbb{C}$ and let F be a topological closed set in $\mathbb{C}$. If $K \cap F = \emptyset$, show that $\operatorname{dist}(K, F) > 0$. On the other hand, construct topological closed sets F_1, F_2 in $\mathbb{C}$ such that $F_1 \cap F_2 = \emptyset$ but $\operatorname{dist}(F_1, F_2) = 0$.

LEMMA 3.3.8. *Let Γ be a simple closed polygonal path (Definition 3.2.1) consists of horizontal lines and vertical lines, such that it contained in a simply connected open set Ω. Let $\{\Gamma_{y_1}, \cdots, \Gamma_{y_n}\}$, for some $y_1 > y_2 > \cdots > y_n$, be all levels of Γ. Let X_1 be the topological closed set in $\mathbb{R}$ such that*

$$\Gamma_{y_0} = \left\{ \ x + \mathbf{i}y_1 \mid x \in X_1 \ \right\}.$$

Then the set $R := \left\{ \ z = x + \mathbf{i}y \mid x \in X_1, y_2 \leq y \leq y_1 \ \right\}$ is contained in Ω.

SKETCH OF PROOF. Note that R is a finite union of disjoint topological closed rectangles. In addition, by using Exercise 3.3.7, we also see that $\delta := \operatorname{dist}(\Gamma, \mathbb{C} \setminus \Omega) > 0$. Let $z_0 \in R$ and let γ be any continuous curve which "connecting z_0 to ∞" in the sense of $\gamma = [\gamma(t) : 0 \leq t < \infty]$ with

$$\gamma(0) = z_0, \quad \lim_{t \to \infty} |\gamma(t)| = \infty.$$

In fact, we have $\gamma \cap \Gamma \neq \emptyset$, this is just simply the fact that, a connected line from R (inside the region bound by Γ) to outside the region bound by Γ, must pass through the boundary. One can refer to [**BN10**, Chapter 8] for those technical details.

We now want to show $z_0 \in \Omega$. Suppose the contrary, that $z_0 \notin \Omega$. Since Ω is simply connected, there exists a continuous curve γ_0 "connected to ∞ by a continuous curve within $\frac{\delta}{2}$-neighborhood of $\mathbb{C} \setminus \Omega$", that is,

$$\operatorname{dist}(\gamma_0(t), \mathbb{C} \setminus \Omega) < \frac{\delta}{2} \text{ for all } t \geq 0, \quad \gamma_0(0) = z_0, \quad \lim_{t \to \infty} |\gamma_0(t)| = \infty.$$

The previous paragraph says that $\gamma_0 \cap \Gamma \neq \emptyset$, then there exists $t_0 \geq 0$ such that $\gamma_0(t_0) \in \Gamma$. From this, we have

$$\operatorname{dist}(\gamma_0(t_0), \mathbb{C} \setminus \Omega) \geq \operatorname{dist}(\Gamma, \mathbb{C} \setminus \Omega) = \delta,$$

which is a contradiction. $\square$

We now generalize the rectangle lemma (Lemma 3.2.8).

LEMMA 3.3.9. *Let f be an analytic function on a simply connected open set Ω, and let Γ be a simple closed polygonal path consists of horizontal lines and vertical lines, which contained in Ω. Then $\int_\Gamma f = 0$.*

SKETCH OF PROOF. We will prove the result by induction on the number of levels. If Γ has only two levels, then Γ is simply the boundary of a closed rectangle, and this case can be concluded by the rectangle lemma (Lemma 3.2.8). The induction step can be done as in the following diagram:

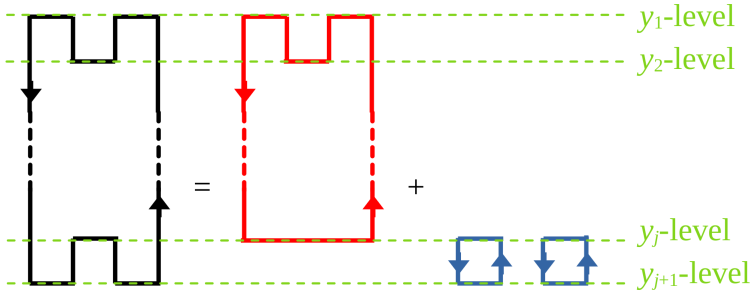


FIGURE 3.3.1. Induction hypothesis for Γ with j-levels (red), and induction step (blue)

Each induction step are done by the rectangle lemma (Lemma 3.2.8). $\square$

From this, we can obtain Fundamental theorem of antiderivative in simply connected domain.

THEOREM 3.3.10 (Fundamental theorem of antiderivative in simply connected domain). *If f is an analytic function on a simply connected open set Ω, then there exists a function F which is analytic and $F' = f$ in Ω. Similarly, such analytic function F is called the (complex)* antiderivative *of f. Combining this with the fundamental theorem of line integral (Theorem 3.1.16), we have*

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}} F' = F(z(b)) - F(z(a)) \tag{3.3.1}$$

for any parametrizable continuous piecewise-C^1 curve $\mathcal{C} = \left[\ z(t) \mid a \leq t \leq b\ \right] \subset \mathcal{R}$.

SKETCH OF PROOF. Choose $z_0 \in \Omega$ and define

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta,$$

where the path of integration is the simple polygonal path consists of horizontal lines and vertical lines, which contained in Ω. This is well-defined by the rectangle lemma (Lemma 3.3.9). Then the rest of proof can be done as in Theorem 3.2.9, which we leave it as an exercise. $\square$

Finally, we state (without proof) the Cauchy closed curve theorem which we needed, which can be proved using Theorem 3.3.10 following the arguments in Theorem 3.2.4. We leave the proof as an exercise.

THEOREM 3.3.11 (Cauchy closed curve theorem in simply connected open set). *Let f be an analytic function on a simply connected open set Ω. For each parametrizable continuous piecewise-C^1 closed curve $\mathcal{C}$ which contained in Ω, then $\int_{\mathcal{C}} f = 0$.*

CHAPTER 4

Properties of Analytic functions

Now we have obtained some fundamental tools connecting the differentiation and integration. We now ready to further study the analytic functions. We first consider the simplest case: the entire functions, which is analytic in the whole $\mathbb{C}$.

4.1. Cauchy integral formula for entire functions

We now try to study the situation stated in Remark 3.2.5. In order to deal with this case, for each point $a \in \mathbb{C}$ and an entire function f, we define the auxiliary function

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & , z \neq a, \\ f'(a) & , z = a. \end{cases} \tag{4.1.1}$$

It is clear that g is continuous. One of the main theme of this section is to prove that g is entire. We first prove the following technical lemma, which sometimes also referred as "rectangle theorem".

Lemma 4.1.1. *Let f be an entire function and let g be the auxiliary function given in (4.1.1). If Γ is the boundary of a topological closed rectangle $\mathcal{R}$, then $\int_\Gamma g = 0$.*

Proof. If $a \notin \overline{\mathcal{R}}$, then clearly g is analytic near $\overline{\mathcal{R}}$, and the lemma immediately follows from Cauchy closed curve theorem (Theorem 3.3.11).

For the case when $a \in \Gamma = \partial\mathcal{R}$, by using Cauchy closed curve theorem (Theorem 3.3.11) one sees that

$$\int_\Gamma g = \int_{\Gamma_1} g,$$

where $\Gamma_1 \ni a$ is the boundary of the square with side length ϵ, as showed in the following figure):

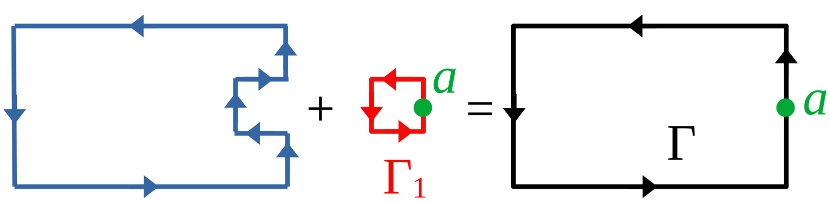


Figure 4.1.1. The sketch of the curves Γ_1 and Γ

By using Lemma 3.1.13, it is easy to see that

$$\left|\int_\Gamma g\right| = \left|\int_{\Gamma_1} g\right| \leq 4\|g\|_{L^\infty(\Gamma_1)}\epsilon \leq 4\|g\|_{L^\infty(\overline{\mathcal{R}})}\epsilon.$$

By arbitrariness of $\epsilon > 0$, we conclude that $\int_\Gamma g = 0$.

For the case when $a \in \operatorname{int}(\mathcal{R})$, by using Cauchy closed curve theorem (Theorem 3.3.11), one sees that

$$\int_{\Gamma} g = \int_{\Gamma_2} g,$$

where Γ_2 is the boundary of the square (which containing a in its interior) with side length ϵ, as showed in the following figure:

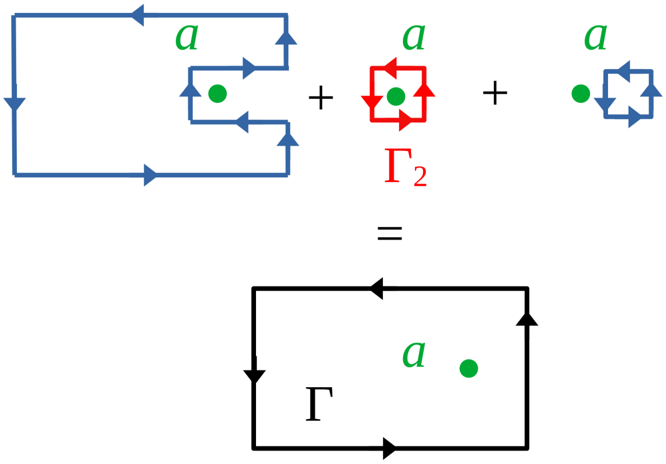


FIGURE 4.1.2. The sketch of the curves Γ_2 and Γ

As in previous case, by using Lemma 3.1.13, it is easy to see that

$$\left| \int_{\Gamma} g \right| = \left| \int_{\Gamma_2} g \right| \leq 4 \|g\|_{L^\infty(\Gamma_2)} \epsilon \leq 4 \|g\|_{L^\infty(\overline{\mathcal{R}})} \epsilon.$$

By arbitrariness of $\epsilon > 0$, we conclude that $\int_{\Gamma} g = 0$. $\square$

The following exercise can be done using similar arguments as in the Fundamental theorem of antiderivative in rectangle (Theorem 3.2.9) and the Cauchy closed theorem in rectangle (Theorem 3.2.4):

EXERCISE 4.1.2. Let $a \in \mathbb{C}$ and let f be an entire function. Show that there exists an entire function G such that $G' = g$, where g is the auxiliary function given in (4.1.1). In addition, one also has $\int_{\mathcal{C}} g = 0$ for all parametrizable continuous piecewise-C^1 closed curve $\mathcal{C}$. [Hint: g is continuous.]

REMARK 4.1.3. Even though we have Lemma 4.1.2, we still don't know whether g is entire or not. At this point, we do not know yet whether the (complex) derivative of entire function is also entire or not.

We now prove the following lemma, which is related to Remark 3.2.5.

LEMMA 4.1.4. *If $\mathcal{C}_\rho(z_0)$ is the boundary of $B_\rho(z_0)$ in counterclockwise orientation, that is,* $\mathcal{C}_\rho(z_0) = \left[\, Re^{\mathbf{i}\theta} + z_0 \,\middle|\, 0 \leq \theta \leq 2\pi \, \right]$, *then*

$$\int_{\mathcal{C}_\rho(z_0)} \frac{1}{z-a} \, \mathrm{d}z = 2\pi\mathbf{i} \quad \textit{for all } a \in B_\rho(z_0).$$

PROOF. We first consider the case when $a = z_0$. In this case, from the definition of line integral (Definition 3.1.4), we see that

$$\int_{\mathcal{C}_\rho(z_0)} \frac{1}{z - z_0}\,\mathrm{d}z = \int_0^{2\pi} \frac{\mathbf{i}Re^{\mathbf{i}\theta}}{Re^{\mathbf{i}\theta}}\,\mathrm{d}\theta = 2\pi\mathbf{i}.$$

By using the fundamental theorem of line integral (Theorem 3.1.16)

$$\int_{\mathcal{C}_\rho(z_0)} \frac{1}{(z - z_0)^2}\,\mathrm{d}z = -\int_{\mathcal{C}_\rho(z_0)} \partial_z \left(\frac{1}{z - z_0}\right)\mathrm{d}z = 0.$$

Inductively, we also see that

$$\int_{\mathcal{C}_\rho(z_0)} \frac{1}{(z - z_0)^{k+1}}\,\mathrm{d}z = -\frac{1}{k}\int_{\mathcal{C}_\rho(z_0)} \partial_z \left(\frac{1}{(z - z_0)^k}\right)\mathrm{d}z = 0 \quad \text{for all } k = 1, 2, \cdots. \tag{4.1.2}$$

We now prove Lemma 4.1.4 for $a \in B_\rho(z_0)$. We write

$$\frac{1}{z-a} = \frac{1}{(z-z_0) - (a-z_0)} = \frac{1}{z-z_0}\cdot\frac{1}{1 - \frac{a-z_0}{z-z_0}} \quad \text{for all } z \in \mathcal{C}_\rho(z_0).$$

Since

$$\left|\frac{a-z_0}{z-z_0}\right| = \frac{|a-z_0|}{\rho} < 1 \quad \text{for all } z \in \mathcal{C}_\rho(z_0), \tag{4.1.3}$$

then the fact that $\frac{1}{1-w} = 1 + w + w^2 + \cdots$ for $w \in \mathbb{C}$ with $|w| < 1$ (geometric sequence), then

$$\frac{1}{z-a} = \frac{1}{z-z_0}\cdot\left(1 + \frac{a-z_0}{z-z_0} + \left(\frac{a-z_0}{z-z_0}\right)^2 + \cdots\right) = \sum_{k=0}^{\infty} \frac{(a-z_0)^k}{(z-z_0)^{k+1}} \quad \text{for all } z \in \mathcal{C}_\rho(z_0). \tag{4.1.4}$$

Again by (4.1.3), we have

$$\begin{aligned}
\limsup_{n\to\infty} \left\| \sum_{k=0}^{n} \frac{(a-z_0)^k}{(z-z_0)^{k+1}} - \frac{1}{z-a} \right\|_{L^\infty(\mathcal{C}_\rho(z_0))} &= \limsup_{n\to\infty} \left\| \sum_{k=n+1}^{\infty} \frac{(a-z_0)^k}{(z-z_0)^{k+1}} \right\|_{L^\infty(\mathcal{C}_\rho(z_0))} \\
\leq \limsup_{n\to\infty} \sum_{k=n+1}^{\infty} \left\| \frac{(a-z_0)^k}{(z-z_0)^{k+1}} \right\|_{L^\infty(\mathcal{C}_\rho(z_0))} &= \frac{1}{\rho}\limsup_{n\to\infty} \sum_{k=n+1}^{\infty} \left(\frac{|a-z_0|}{\rho}\right)^k = 0,
\end{aligned}$$

that is, the convergence in (4.1.4) is uniform. Therefore, from (4.1.2) we obtain

$$\int_{\mathcal{C}_\rho(z_0)} \frac{1}{z-a}\,\mathrm{d}z = \int_{\mathcal{C}_\rho(z_0)} \frac{1}{z-z_0}\,\mathrm{d}z + \sum_{k=1}^{\infty}(a-z_0)^k \int_{\mathcal{C}_\rho(z_0)} \frac{1}{(z-z_0)^{k+1}}\,\mathrm{d}z = 2\pi\mathbf{i},$$

which conclude our lemma. □

Warning: In general the infinite sum and integral are not commute. The uniform convergence is a sufficient condition that guarantees that this idea work.

EXERCISE 4.1.5. Prove (4.1.2) by direct evaluation in the definition of line integral (Definition 3.1.4).

We now ready to state and proof the main theorem of this section.

THEOREM 4.1.6 (Cauchy integral formula for entire functions). *Let f be an entire function, let $a \in \mathbb{C}$ and let $\mathcal{C} = \left[\ Re^{\mathbf{i}\theta} \,\middle|\, 0 \leq \theta \leq 2\pi\ \right]$ with $R > |a|$. Then*

$$f(a) = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)}{z-a} \,\mathrm{d}z.$$

PROOF. By Exercise 4.1.2 and Lemma 4.1.4, one has

$$0 = \int_{\mathcal{C}} \frac{f(z)-f(a)}{z-a} \,\mathrm{d}z = \int_{\mathcal{C}} \frac{f(z)}{z-a} \,\mathrm{d}z - f(a) \int_{\mathcal{C}} \frac{f(z)}{z-a} \,\mathrm{d}z = \int_{\mathcal{C}} \frac{f(z)}{z-a} \,\mathrm{d}z - 2\pi\mathbf{i} f(a),$$

which conclude our theorem. □

4.2. Power series (with $R = \infty$) and entire function

In Chapter 2 we have showed that each power series represents an analytic function inside its domain of convergence. In real analysis, it is known that there exists a C^∞ function such that its Taylor expansion does not converges to it. For example, we consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & , x > 0, \\ 0 & , x \leq 0, \end{cases}$$

which is in $C^\infty(\mathbb{R})$ but $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$ (so that its Taylor expansion at 0 vanishes identically, therefore does not converge to f). In other words, the differentiability (existence of partial derivatives) does not guarantee the convergence of Taylor sequence. However, the complex differentiation has the following surprising properties, which is the main result of this section:

THEOREM 4.2.1. *f is entire if and only if it has a power series representation (centered at some $a \in \mathbb{C}$ with radius of convergence $= \infty$). In this case, for each $a \in \mathbb{C}$, the complex derivatives $\{f^{(k)}(a)\}_{k=1}^\infty$ exist and satisfies*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k \quad \text{for all } z \in \mathbb{C}. \tag{4.2.1}$$

REMARK 4.2.2. The above theorem means that $\{f^{(k)}(a)\}_{k=1}^\infty$ exist for all $a \in \mathbb{C}$, that is, f is infinitely complex differentiable.

THEOREM 4.2.1. If f has a power series representation at $a \in \mathbb{C}$ with radius of convergence $= \infty$, i.e. there exist $C_k \in \mathbb{C}$ such that $f(z) = \sum_{k=0}^\infty C_k (z-a)^k$ for all $z \in \mathbb{C}$. By applying Theorem 2.2.9 $g(z) = f(z+a) = \sum_{k=0}^\infty C_k z^k$, we know that g is entire, and so is f.

Conversely, we now suppose that f is entire. Given any $a \in \mathbb{C}$, we define the entire function $g(z) := f(z+a)$ for all $z \in \mathbb{C}$. If we can show that

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} z^k \text{ for all } z \in \mathbb{C}, \tag{4.2.2}$$

then $f(z) = g(z-a) = \sum_{k=0}^\infty \frac{g^{(k)}(0)}{k!}(z-a)^k = \sum_{k=0}^\infty \frac{f^{(k)}(a)}{k!}(z-a)^k$, which conclude (4.2.1).

It is remain to prove (4.2.2). Given any $z \in \mathbb{C}$, one can choose $R > 0$ such that $|z| < R$. By using the Cauchy integral formula for entire function, one has

$$g(y) = \frac{1}{2\pi\mathbf{i}} \int_{\partial B_R} \frac{g(w)}{w-y} \,\mathrm{d}w \quad \text{for all } y \in B_R.$$

By using the geometric sequence (which used in the proof of Lemma 4.1.4), one sees that

$$\frac{1}{w-y} = \frac{1}{w(1-\frac{y}{w})} = \frac{1}{w} + \frac{y}{w^2} + \frac{y^2}{w^3} + \cdots = \sum_{k=0}^{\infty} \frac{y^k}{w^{k+1}}$$

which is uniformly converge, so that

$$g(y) = \sum_{k=0}^{\infty} \frac{1}{2\pi\mathbf{i}} \left(\int_{\partial B_R} \frac{g(w)}{w^{k+1}} \,\mathrm{d}w \right) y^k \quad \text{for all } y \in B_R.$$

Then by Exercise 2.2.10, one reach

$$\frac{1}{2\pi\mathbf{i}} \left(\int_{\partial B_R} \frac{g(w)}{w^{k+1}} \,\mathrm{d}w \right) = \frac{g^{(k)}(0)}{k!} \quad \text{for all } k = 0, 1, 2, \cdots$$

and hence

$$g(y) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} y^k \quad \text{for all } y \in B_R.$$

Since $z \in B_R$, then

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} z^k.$$

Since the above procedure holds true for all $z \in \mathbb{C}$, hence we conclude (4.2.2). □

EXERCISE 4.2.3 (Higher order Cauchy integral formula for entire functions). Let f be an entire function, let $a \in \mathbb{C}$ and let $\mathcal{C} = \left[\, Re^{\mathbf{i}\theta} \,\middle|\, 0 \leq \theta \leq 2\pi \, \right]$ with $R > |a|$. Show that

$$f^{(k)}(a) = \frac{k!}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)}{(z-a)^{k+1}} \,\mathrm{d}z \quad \text{for all } k = 0, 1, 2, \cdots$$

PROPOSITION 4.2.4. *If f is entire, then the auxiliary function g given in (4.1.1) is also entire.*

PROOF. We can write (4.2.1) as

$$f(z) - f(a) = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k \quad \text{for all } z \in \mathbb{C},$$

where we choose $a \in \mathbb{C}$ be the number as in (4.1.1). Dividing the above equation by $(z-a)$, we reach

$$g(z) \equiv \frac{f(z) - f(a)}{z-a} = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^{k-1} = \sum_{m=0}^{\infty} \frac{f^{(m+1)}(a)}{(m+1)!} (z-a)^m \quad \text{for all } z \neq a.$$

Since g is continuous on $\mathbb{C}$, and the right hand side of the above inequality is entire (hence continuous), thus the above identity also holds for all $z \in \mathbb{C}$, which completes the proof. □

EXERCISE 4.2.5. Suppose that f is entire with zeros $a_1, a_2, \cdots a_N$, that is, $f(a_k) = 0$ for $k = 1, 2, \cdots, N$, and we define

$$g(z) := \frac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_N)} \quad \text{for all } z \in \mathbb{C} \setminus \{a_1, a_2, \cdots, a_N\}.$$

Show that if $\lim_{z\to a_k} g(z)$ exists for all $k = 1, 2, \cdots, N$, then the extension $\tilde{g}$ of g defined by

$$\tilde{g}(z) := \begin{cases} g(z) & , z \in \mathbb{C} \setminus \{a_1, a_2, \cdots, a_N\}, \\ \lim\limits_{z\to a_k} g(z) & , z = a_k \text{ for } k = 1, 2, \cdots, N, \end{cases}$$

is also entire.

4.3. Liouville theorem and the fundamental theorem of algebra

By using the Cauchy integral formula for entire functions, we also can obtain some powerful tools, which are well-known.

Theorem 4.3.1 (Liouville theorem). *A bounded entire function is constant.*

Proof. Let a and b represent any two complex numbers and let C be any positively oriented (i.e. counter clockwise oriented) circle centered at 0 and with radius $R > \max\{|a|, |b|\}$. By using the Cauchy integral formula for entire functions (Theorem 4.1.6), we see that

$$f(b) - f(a) = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)}{z-b}\,\mathrm{d}z - \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)}{z-a}\,\mathrm{d}z = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)(b-a)}{(z-a)(z-b)}\,\mathrm{d}z.$$

Since the arc length $\mathscr{H}^1(\mathcal{C})$ of $\mathcal{C}$ is $2\pi R$, then

$$|f(b) - f(a)| \le \frac{1}{2\pi} \frac{\|f\|_{L^\infty(\mathcal{C})}|b-a|}{(R-|a|)(R-|b|)} \mathscr{H}^1(\mathcal{C}) = \frac{\|f\|_{L^\infty(\mathcal{C})}|b-a|}{(R-|a|)(R-|b|)} R.$$

Taking $R \to \infty$ (in the sense of limit supremum), we conclude $f(a) = f(b)$. Since a, b are arbitrary, then we conclude our theorem. □

Theorem 4.3.2 (Extended Liouville theorem). *Let $A > 0$, $B > 0$ and $k \in \mathbb{Z}_{\ge 0}$. If the entire function f satisfies*

$$|f(z)| \le A + B|z|^k \quad \text{for all } z \in \mathbb{C}, \tag{4.3.1}$$

then f is an analytic polynomial of degree at most k.

Proof. We prove the above result by induction on k. The statement for $k = 0$ is just simply Theorem 4.3.1.

It is suffice to prove the result for $k = \ell + 1$ if Theorem 4.3.2 holds true for $k = \ell \ge 0$. Let g be the auxiliary function given in 4.1.1 and choosing $a = 0$. From Proposition 4.2.4 we know that such g is entire. We also see that

$$|g(z)| = \frac{|f(z) - f(0)|}{|z|} \le \frac{|f(z)| + |f(0)|}{|z|} \le \frac{2A + B|z|^{\ell+1}}{|z|} \le 2A + B|z|^\ell \quad \text{for all } |z| \ge 1,$$

and thus

$$|g(z)| \le \|g\|_{L^\infty(B_1)} + 2A + B|z|^\ell.$$

By using the induction hypothesis that Theorem 4.3.2 holds true for $k = \ell \ge 0$, we know that g is an analytic polynomial of degree at most ℓ. Since

$$f(z) = zg(z) + f(0) \quad \text{for all } z \ne 0,$$

by analyticity of both f and g, in particular the above identity also holds true for all $z \in \mathbb{C}$. Therefore f is analytic polynomial of degree at most $\ell + 1$. This conclude Theorem 4.3.2 by induction. □

EXERCISE 4.3.3. Suppose f is entire and $|f(z)| \leq A + B|z|^{\frac{3}{2}}$ for all $z \in \mathbb{C}$. Show that f is linear polynomial.

EXERCISE 4.3.4. Suppose f is entire and $|f'(z)| \leq |z|$ for all $z \in \mathbb{C}$. Show that $f(z) = a + bz^2$ with $|b| \leq \frac{1}{2}$.

LEMMA 4.3.5. *Let $P(z)$ be a analytic polynomial which is not identical to a constant function. Then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.*

PROOF. Suppose the contrary that such $z_0 \in \mathbb{C}$ does not exist, that is, $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then by Lemma 2.1.5 one sees that $f(z) := \frac{1}{P(z)}$ is an entire function. Since P is non-constant, then we can write

$$P(z) = \sum_{j=0}^{N} c_j z^j$$

for some $N \in \mathbb{N}$ with $c_N \neq 0$ and $c_n = 0$ for all $n > N$. Then we see that

$$\liminf_{z\to\infty} |P(z)| \geq \liminf_{z\to\infty} \left(|c_N||z|^N - \sum_{j=0}^{N-1} |c_j||z|^j \right) = \infty,$$

which shows that

$$\lim_{z\to\infty} |f(z)| = 0.$$

Therefore f is a bounded entire function, which is a constant by Liouville theorem (Theorem 4.3.1), this shows that P must identical to a constant function, which is a contradiction. □

We finally end this section by proving an important theorem in the field theory.

THEOREM 4.3.6 (Fundamental theorem of algebra). *Let $P(z)$ be a analytic polynomial which is not identical to a constant function, then there exists $A, \alpha_1, \cdots, \alpha_N \in \mathbb{C}$ such that $P(z) = A(z-\alpha_1)\cdots(z-\alpha_N)$ for all $z \in \mathbb{C}$. In other words, the complex field $\mathbb{C}$ is* algebraically complete*.*

PROOF. Write $P(z) = \sum_{j=0}^{N} c_j z^j$ for some $N \in \mathbb{N}$ with $c_N \neq 0$. Similar in the proof of the extended Liouville theorem (Theorem 4.3.2), we see that the auxiliary function g given in 4.1.1 and choosing $a = \alpha$ satisfies

$$|g(z)| \leq A + B|z|^{N-1},$$

and hence by the extended Liouville theorem (Theorem 4.3.2), g must be an analytic polynomial. Again, similar in the proof of the extended Liouville theorem (Theorem 4.3.2), we have

$$P(z) = g(z)(z - \alpha) \quad \text{for all } z \in \mathbb{C},$$

this shows that g must be a polynomial of degree $N - 1$. Repeating the above arguments on g, we conclude our theorem. □

4.4. The roots of ± 1

We now include some materials from [**FB09**]. In the very beginning of this course, we asked a question regarding how to define $\sqrt{-1}$. By using the fundamental theorem of algebra (Theorem 4.3.6), we now know that the equation $z^2+1=0$ has exactly two solutions in $\mathbb{C}$, and they are $\pm\mathbf{i}$. As a corollary, we note that

$$\text{the equation } z^2+1=0 \text{ has no roots in } \mathbb{R}.$$

Therefore, the polynomial $P(z)=z^2+1$ is *irreducible* in $\mathbb{R}[z]$. For convenience, we usually write $\sqrt{-1} := \mathbf{i}$, but one should be aware that $\sqrt{-1}$ is not well-defined as a function in general. In complex analysis, we call $-\mathbf{i}$ is another *branch* of $\sqrt{-1}$.

It is well-known that the n-root of 1 is well-defined in $\mathbb{R}$, which is given by $\sqrt[n]{1}=1$. However, in complex field, we have the following interesting observation (one also asks similar questions in finite field):

THEOREM 4.4.1. *For each $n\in\mathbb{N}$, there are exactly n different solutions $\{\zeta_j\}_{j=1}^n$ (or roots) of $z^n-1=0$, and they have the formula*

$$\zeta_j = \cos\frac{2\pi j}{n} + \mathbf{i}\sin\frac{2\pi j}{n} \quad \text{for } j=0,1,2,\cdots,n-1. \tag{4.4.1}$$

We called (4.4.1) the n^{th} roots of unity. *We also called z^n-1 the* cyclotomic equation, *since (4.4.1) is exactly the vertex of regular n-gon in $\mathbb{C}$*

PROOF. By using Exercise 2.3.2, one can directly verify that (4.4.1) are n different roots of $z^n-1=0$. By using the fundamental theorem of algebra (Theorem 4.3.6), they are exactly all the n different solutions. $\square$

EXERCISE 4.4.2 (n-roots of -1). For each integer $n\geq 2$, determine all roots of the equation $z^n+1=0$.

4.5. Cauchy integral formula in a ball

We have proved the Cauchy integral formula for entire functions in Section 4.1. By carefully inspecting the arguments, in fact we can obtain a local version. Here we will exhibit the details.

Let f be an analytic function in a ball $B_r(z_0)$. By using the fundamental theorem of antiderivative in rectangle (see Theorem 3.2.9 and (3.2.3)), one sees that the function

$$F(z)=\int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta \equiv \int_{\mathcal{C}} f(\zeta)\,\mathrm{d}\zeta \text{ is analytic and satisfies } F'=f \text{ on } B_r(z_0),$$

where $\mathcal{C}$ denotes the oriented curve consists of the straight lines from z_0 to $z_0+\mathfrak{Re}\,(z-z_0)$ and then from $z_0+\mathfrak{Re}\,(z-z_0)$ to z. It is important to notice that one can find a topological closed rectangle consists of z_0 and z which is contained in $B_r(z_0)$.

We consider the auxiliary function g similar to (4.1.1): If f is analytic in $B_r(z_0)$ and $a\in B_r(z_0)$, then we define the function

$$g(z)=\begin{cases} \frac{f(z)-f(a)}{z-a} & ,z\in B_r(z_0)\setminus\{a\}, \\ f'(a) & ,z=a, \end{cases} \tag{4.5.1}$$

which is continuous on $B_r(z_0)$. At this moment, we don't know whether g is analytic in D yet. However, by continuity of g and following the same arguments as in Exercise 4.1.2, one can show that

$$\text{there exists an analytic function } G \text{ with } G' = g \text{ on } B_r(z_0). \tag{4.5.2}$$

In addition, one also has

$$\int_{\mathcal{C}} g = 0 \text{ for all parametrizable continuous piecewise-}C^1 \text{ closed curve } \mathcal{C} \subset B_r(z_0). \tag{4.5.3}$$

We now can easily proof the local version of Cauchy integral formula.

THEOREM 4.5.1 (Cauchy integral formula in a ball). *Suppose that f is analytic in $B_r(z_0)$ and let $a \in B_r(z_0)$. For each $0 < \rho < r$ with $a \in B_\rho(z_0)$, one has*

$$f(a) = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}_\rho(z_0)} \frac{f(\omega)}{\omega - a} \,\mathrm{d}\omega,$$

where $\mathcal{C}_\rho(z_0)$ is the closed curve $\mathcal{C}_\rho(z_0) = \left[\ z_0 + \rho e^{\mathbf{i}\theta} \,\middle|\, 0 \leq \theta \leq 2\pi\ \right]$, that is, $\mathcal{C}_\rho(z_0) = \partial B_\rho(z_0)$ with counterclockwise oriented.

PROOF. Let g be the auxiliary function given in (4.5.1). By using (4.5.3) and Lemma 4.1.4, one has

$$0 = \int_{\mathcal{C}_\rho(z_0)} \frac{f(\omega) - f(a)}{\omega - a} \,\mathrm{d}\omega = \int_{\mathcal{C}_\rho(z_0)} \frac{f(\omega)}{\omega - a} \,\mathrm{d}\omega - \int_{\mathcal{C}_\rho(z_0)} \frac{f(a)}{\omega - a} \,\mathrm{d}\omega = \int_{\mathcal{C}_\rho(z_0)} \frac{f(\omega)}{\omega - a} \,\mathrm{d}\omega - 2\pi\mathbf{i} f(a),$$

which conclude our theorem. □

EXERCISE 4.5.2. Let Ω be an open set, let f be an analytic function on Ω and let $a \in \Omega$. Show that $f(a)$ is equal to the mean value of f takes around the boundary of any disc centered at a contained in D, that is,

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{\mathbf{i}\theta}) \,\mathrm{d}\theta$$

whenever $\partial B_r(a) \subset D$.

REMARK 4.5.3. As we see in Remark 2.1.11, an analytic function always a harmonic function. In fact, the mean value theorem also holds true for harmonic function, see [**GT01**]. This even holds true for Helmholtz operator $\Delta + k^2$, see e.g. my work [**KLSS22**, Appendix].

4.6. Power series (with $R < \infty$) and analytic function

In Chapter 2 we have showed that each power series represents an analytic function inside its domain of convergence. We denote R be its radius of convergence. In Section 4.2 we have showed the converve of this theorem for the case when $R = \infty$. We now turn to the question about the case when $R < \infty$.

THEOREM 4.6.1. *If f is analytic in $B_R(z_0)$, there exist constants C_k such that*

$$f(z) = \sum_{k=0}^{\infty} C_k (z - z_0)^k \quad \textit{for all } z \in B_R(z_0).$$

PROOF. For each $0 < \rho < R$, by using the Cauchy integral formula in a ball (Theorem 4.5.1) with $a = z$, we have (Theorem 4.5.1)

$$f(z) = \frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_\rho(z_0)} \frac{f(\omega)}{\omega - z} \, d\omega \quad \text{for all } z \in B_\rho(z_0).$$

Recall (4.1.4) and changing the notation $z \to \omega$ and $a \to z$:

$$\frac{1}{\omega - z} = \frac{1}{\omega - z_0} \cdot \left(1 + \frac{z - z_0}{\omega - z_0} + \left(\frac{z - z_0}{\omega - z_0} \right)^2 + \cdots \right) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\omega - z_0)^{k+1}} \quad \text{for all } \omega \in \mathcal{C}_\rho(z_0),$$

which converges uniformly on $\mathcal{C}_\rho(z_0)$. Combining the above two equations, we reach

$$f(z) = \frac{1}{2\pi \mathbf{i}} \sum_{k=0}^{\infty} \left(\int_{\mathcal{C}_\rho(z_0)} \frac{f(\omega)}{(\omega - z_0)^{k+1}} \, d\omega \right) (z - z_0)^k.$$

Arguing as in Theorem 4.2.1 (which involving Exercise 2.2.10), we again have

$$\frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_\rho(z_0)} \frac{f(\omega)}{(\omega - z_0)^{k+1}} \, d\omega = \frac{f^{(k)}(z_0)}{k!},$$

and thus

$$f(z) = \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad \text{for all } z \in B_\rho(z_0).$$

Since $0 < \rho < R$ is arbitrary, then we conclude our theorem. □

From Theorem 4.6.1, we immediately conclude the following corollary.

COROLLARY 4.6.2 (**Local** power series representation). *Let Ω be an open set in $\mathbb{C}$. Then f is analytic in Ω if and only if it has a local power series near each point in Ω, i.e. for each $z_0 \in \Omega$ we can write f as*

$$f(z) = \sum_{k=0}^{\infty} C_k (z - z_0)^k$$

for all $z \in B_R(z_0)$, where $R = \sup_{B_r(z_0) \subset \Omega} r$. In this case, the complex derivatives $\{f^{(k)}(z_0)\}_{k=1}^{\infty}$ exist and satisfies

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \tag{4.6.1}$$

for all $z \in B_R(z_0)$, where $R = \sup_{B_r(z_0) \subset \Omega} r$.

REMARK 4.6.3. One sees that Theorem 4.2.1 is just a special case $\Omega = \mathbb{C}$ of Corollary 4.6.2. One should aware that the power series (4.6.1) in general not holds for all $z \in D$, i.e. not global! See Remark 2.2.5. This is the reason why we called (4.6.1) the **local** power series.

PROPOSITION 4.6.4. *If f is analytic near a, then so is the auxiliary function g given in* (4.5.1).

PROOF. By using Corollary 4.6.2, we see that

$$f(z) - f(a) = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k \quad \text{for all } z \text{ near } a.$$

and thus

$$g(z) = \frac{f(z) - f(a)}{z-a} = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^{k-1} = \sum_{\ell=0}^{\infty} \frac{f^{(\ell+1)}(a)}{(\ell+1)!}(z-a)^{\ell} \quad \text{for all } z \neq a \text{ near } a.$$

By continuity of g, we see that the above identity also holds true for $z = a$, which conclude our proposition. $\square$

THEOREM 4.6.5 (Uniqueness continuation property). *Let f be an analytic function on an open connected set Ω. If there exists a nonempty open set $D \subset \Omega$ such that $f|_D = 0$, then $f \equiv 0$ in Ω.*

REMARK 4.6.6. By using the Carleman estimate, this property can be extended to large class of solution of elliptic equations and systems (recall that analytic function also harmonic, see also Remark 2.1.11). A related problem is called the *Landis conjecture*, which can be referred as the *unique continuation property from infinity*.

PROOF OF THEOREM 4.6.5. We will prove this using a standard argument for open connected set in Remark 1.2.19. We define

$$A := \left\{ z_0 \in \Omega \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{z_n\} \subset \Omega \text{ such that} \\ z_n \to z_0 \text{ and } f(z_n) = 0 \text{ for all } n \in \mathbb{N} \end{array} \right\}.$$

Since f is contiuous, one sees that

$$f(z) = 0 \text{ if and only if } z \in A. \tag{4.6.2}$$

We first show that A is open (in $\mathbb{C}$ iff relative to Ω, since Ω is open, see Remark 1.2.15). Let $z_0 \in A$. By Corollary 4.6.2, one can represent f using a local power series near z_0, that is, there exists $\epsilon > 0$ such that $f(z) = \sum_k C_k(z-z_0)^k$ for all $z \in B_\epsilon(z_0)$. Then by the uniqueness theorem of power series (Theorem 2.2.11) we see that $f = 0$ in $B_\epsilon(z_0)$, and hence $B_\epsilon(z_0) \subset A$. By arbitrariness of $z_0 \in A$, we conclude that A is open.

On the other hand, we want to show that $\Omega \setminus A$ is open as well. We first see that (4.6.2) is equivalent to

$$f(z) \neq 0 \iff z \in \Omega \setminus A.$$

Given any $z_0 \in \Omega \setminus A$, we have $f(z_0) \neq 0$. We now choose $\epsilon = \frac{1}{2}|f(z_0)| > 0$. By continuity of f at z_0, there exists $\delta > 0$ such that

$$w \in B_\delta(z_0) \implies |f(w) - f(z_0)| \leq \epsilon = \frac{1}{2}|f(z_0)|.$$

This gives

$$\begin{aligned} & w \in B_\delta(z_0) \\ & \implies |f(z_0)| - |f(w)| \leq |f(w) - f(z_0)| \leq \frac{1}{2}|f(z_0)| \\ & \implies \frac{1}{2}|f(z_0)| \leq |f(w)| \\ & \implies f(w) \neq 0 \implies w \in \Omega \setminus A. \end{aligned}$$

Hence we see that $B_\delta(z_0) \subset \Omega \setminus A$. By arbitrariness of $z_0 \in \Omega \setminus A$, this shows that $\Omega \setminus A$ is open.

Since both A and $\Omega \setminus A$ are open, by connectness of Ω, we see that either $\Omega = \emptyset$ or $\Omega = A$. Since $\Omega \supset D \neq \emptyset$, we finally conclude that $\Omega = A$, which conclude our theorem. $\square$

COROLLARY 4.6.7 (Uniqueness theorem). *Let f be an analytic function on an open connected set Ω. If there exists a sequence $\{z_n\} \subset \Omega$ such that $z_n \to z_0 \in \Omega$ and $f(z_n) = 0$ for all $n \in \mathbb{N}$, then $f \equiv 0$ in Ω.*

PROOF. By using Corollary 4.6.2, one can represent f using a local power series near z_0. By using the uniqueness theorem of power series (Theorem 2.2.11), one sees that there exists $r > 0$ such that $f|_{B_r(z_0)} = 0$. Hence our result immediately follows from the unique continuation property of analytic function (Theorem 4.6.5). $\square$

EXAMPLE 4.6.8. We consider $f(z) = \sin z$, which is entire (i.e. analytic in $\Omega = \mathbb{C}$). One sees that f has at least infinitely many zeros: $f(n\pi) = 0$ for all $n \in \mathbb{Z}$. These zeros does not converge in $\mathbb{C}$. In fact, by using Corollary 4.6.7, the set of zeros of f does not have a limit point. Therefore, given any bounded set, it contains at most finitely many zeros of f.

EXAMPLE 4.6.9. We consider $f(z) = \sin(\frac{1}{z})$, which is analytic in $\Omega = \mathbb{C} \setminus \{0\}$. One sees that f has infinitely many zeros: $f(\frac{1}{n\pi}) = 0$ for all $n \in \mathbb{Z}$, and these zeros converge at 0. This illustrate the analyticity assumption in Corollary 4.6.7 is essential.

THEOREM 4.6.10. *If f is entire and if $|f(z)| \to \infty$ as $z \to \infty$, then f is a polynomial.*

PROOF. By hypothesis, there exists $R > 0$ such that $|f(z)| > 1$ for all $|z| > R$. This shows that f cannot have any zeros outside $B_R(0)$, and hence there at most finitely many zeros in $\overline{B_R(0)}$. If not, by using Bolzano-Weierstrass theorem, there exists a sequence $\{z_n\} \subset \overline{B_R(0)}$ converges to $z \in \overline{B_R(0)}$ with $f(z_n) = 0$. Hence the uniqueness theorem in Corollary 4.6.7 (with $\Omega = \mathbb{C}$) implies that $f \equiv 0$ throughout $\mathbb{C}$, which is a contradiction.

We now denote $\alpha_1, \cdots, \alpha_N \in B_R(0)$ be the zeros of f (it is possible that $\alpha_i = \alpha_j$ for some $i \neq j$). By using Exercise 4.2.5, we see that the function

$$g(z) := \frac{f(z)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_N)}$$

is entire and also $g(z) \neq 0$ for all $z \in \mathbb{C}$. Hence we see that

$$h(z) := \frac{1}{g(z)} = \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_N)}{f(z)}$$

is also entire. Since $|f(z)| \to \infty$ as $z \to \infty$, then $|h(z)| \leq A + |z|^N$. By using the extended Liouville theorem (Theorem 4.3.2), we see that h is a polynomial. But however $h(z) = \frac{1}{g(z)} \neq 0$ for all $z \in \mathbb{C}$, then by fundamental theorem of algebra (Theorem 4.3.6), we conclude that h is a constant function, says $h(z) = k$ for some constant $k \neq 0$. By the definition of h, we see that

$$f(z) = \frac{1}{k}(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_N),$$

which conclude our theorem. $\square$

4.7. Morera Theorems

The key result in our discussion of analytic functions so far has been the Cauchy closed curve theorem (Theorem 3.3.11). In fact, the partial converse holds true as below (for future convenience, we will refer all theorems in this section the "Morera theorems"):

THEOREM 4.7.1 (Morera). *Let f be a* continuous *function in an open set Ω. If*

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0$$

for all Γ the boundary of topological closed rectangle in Ω, each segment is either horizontal (i.e. parallel to real axis) or vertical (i.e. parallel to imaginary axis), then f is analytic in Ω.

REMARK 4.7.2. In view of the Cauchy integral formula (Theorem 4.5.1), one sees that the continuity of f is a necessary hypothesis.

EXERCISE 4.7.3. Prove Theorem 4.7.1 by modifying the arguments in the fundamental theorem of antiderivative in rectangle (Theorem 3.2.9).

Morera's theorem is often used to establish the analyticity of functions given in integral form.

EXERCISE 4.7.4. Using Morera's theorem and Fubini's theorem (carefully check the sufficient conditions for Fubini Theorem!) to show that the function $f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\,\mathrm{d}t$ is analytic in the left half plane $\left\{\ z \in \mathbb{C} \;\middle|\; \mathfrak{Re}\,(z) < 0\ \right\}$.

THEOREM 4.7.5 (Morera's uniform convergence theorem). *Suppose $\{f_n\}$ represents a sequence of analytic functions on an open set Ω satisfies*

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty(K)} = 0 \quad \textit{for all compact set } K \subset \Omega,$$

then f is analytic in Ω.

PROOF. Given any $z \in \Omega$, there exists $r > 0$ such that $B_r(z) \subset \Omega$. We choose the compact set $K = \overline{B_{\frac{r}{2}}(z)}$. Hence we have

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty(B_{\frac{r}{2}}(z))} = 0.$$

This shows that f is continuous on K. Furthermore, [Morera]for each Γ the boundary of any topological closed rectangle in K, the uniform convergence of f_n to f (on Γ) guarantees that

$$\int_\Gamma f = \lim_{n\to\infty} \int_\Gamma f_n = 0,$$

where the second identity is just simplyby the Cauchy closed curve theorem (Theorem 3.3.11). By Morera's theorem, we conclude that f is analytic in $B_{\frac{r}{2}}(z)$. By arbitrariness of $z \in \Omega$, we conclude the theorem. □

EXERCISE 4.7.6. Show that $g(z) = z_0 + e^{\mathbf{i}\theta} z$ with $\theta = \arg(z_1 - z_0)$, maps the real axis $\left\{\ z \in \mathbb{C} \;\middle|\; \mathfrak{Im}\, z = 0\ \right\}$ onto the line L through z_0 and z_1. Here $\arg w$ is defined (modulo 2π) as that number θ for which

$$\sin\theta = \frac{\mathfrak{Im}\, w}{|w|}, \quad \cos\theta = \frac{\mathfrak{Re}\, w}{|w|}.$$

Clearly, g defines an entire function.

THEOREM 4.7.7 (Morera's continuity theorem). *Let Ω be an open set and let L be a straight line in $\mathbb{C}$. If f is continuous in Ω and analytic in $\Omega \setminus L$, then f is analytic in Ω.*

PROOF. By using Exercise 4.7.6, it is suffice to show the theorem when L is the real axis. Let $z_0 \in L \cap \Omega$, and let $r > 0$ be such that $B_r(z_0) \subset \Omega$. Let Γ the boundary of any topological closed rectangle in $B_r(z_0)$ which are parallel to the real and imaginary axes.

Case 1: L does not meet the topological closed rectangle enclosed by Γ. In this case, f is analytic near the topological closed rectangle and thus $\int_\Gamma f = 0$ by Cauchy closed curve theorem (Theorem 3.3.11).

Case 2: the bottom side of Γ touches L. Let $\epsilon > 0$ sufficiently small and let Γ_ϵ be the rectangle composed of the sides of Γ with bottom side shifted up by ϵ. By the continuity of f, we see that

$$\int_\Gamma f = \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} f = 0,$$

where the second identity follows by the Cauchy closed curve theorem (Theorem 3.3.11).

Case 3: the top side of Γ coincides with L. We can treat this case similar as previous case.

Case 4: The line L pass through the interior of the rectangle enclosed by Γ. In this case, we can divide the rectangle into two rectangle by L. Let Γ_1 and Γ_2 are boundary of these two rectangles. By using Case 2 and Case 3, we see that $\int_{\Gamma_1} f = 0$ and $\int_{\Gamma_2} f = 0$, and hence $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f = 0$.

Putting these 4 cases together, we conclude that f is analytic in $B_r(z_0)$. By arbitrariness of $z_0 \in L$, we conclude our theorem. $\square$

CHAPTER 5

Laurent series and the Cauchy residual theorem

5.1. Riemann's principle of removable singularities

In Remark 3.2.5, we posting the question about what we get if we integral over a simple closed curve which surrounding some singularity. We have encounter some singularities in the Cauchy integral formula (Theorem 4.5.1). Before studying the singularities, let us first classify the singularities. Then we can at least partially answer this question for some class of singularities (so that make this course easier).

DEFINITION 5.1.1. We call the set $B_R(z_0) \setminus \{z_0\}$ the *punctured ball* centered at z_0 with radius R (or called the *deleted neighborhood*). A function f is said to have an *isolated singularity* at z_0 if f is analytic in a punctured ball centered at z_0 and f is not complex differentiable (in the sense of Definition 2.1.1) at z_0.

REMARK 5.1.2. By using Theorem 4.7.7, we see that z_0 is an isolated singularity if and only if f discontinuous at z_0.

DEFINITION 5.1.3. Suppose f has an isolated singularity at z_0.

(1) If there exists a function g, analytic near z_0, such that $f(z) = g(z)$ in a punctured ball centered at z_0, we say that f has a *removable singularity* at z_0.
(2) If there exist functions A and B, both analytic near z_0 with $A(z_0) \neq 0$ and $B(z_0) = 0$, such that $f(z) = \frac{A(z)}{B(z)}$ in a punctured ball centered at z_0, then we say that f has a *pole* at z_0.
(3) If f has neither a removable singularity nor a pole at z_0, we say f has an *essential singularity* at z_0.

In next section, we will fully characterize (necessary and sufficient condition) in next section (Theorem 5.2.6) in terms of Laurent series. In plain words, removable singularity is the one we can basically ignored, while essential singularity is the one that too difficult to handle within this chapter. The pole is the one we want to discuss in this chapter. In this section, we first study some sufficient conditions.

LEMMA 5.1.4 (Riemann's principle of removable singularities). *If f is analytic in a punctured ball centered at z_0 and that $\lim_{z\to z_0}(z-z_0)f(z) = 0$, then f has at most a removable singularity at z_0, i.e. there exists a function A, analytic near z_0, such that $A = f$ in a punctured ball centered at z_0.*

PROOF OF LEMMA 5.1.4. If f is continuous at z_0, then by Theorem 4.7.7 we know that f is analytic near z_0, and we have nothing to proof. If f is discontinuous at z_0, then z_0 is an isolated singularity of f. It is easy to see that the function

$$h(z) = \begin{cases} (z-z_0)f(z) & , z \neq z_0, \\ 0 & , z = z_0, \end{cases}$$

is continuous at z_0. By using Theorem 4.7.7, we see that h is analytic near z_0. Since $h(z_0) = 0$, then the function $A(z) = \frac{h(z)}{z-z_0}$ is analytic near z_0 (see Exercise 4.2.5). Since $A = f$ in a punctured ball centered at z_0, then we conclude our lemma. $\square$

REMARK 5.1.5. If f is analytic and bounded in a punctured ball centered at z_0, then clearly $\lim_{z\to z_0}(z-z_0)f(z) = 0$, and thus the above lemma follows that f has (at most) a removable singularity at z_0.

REMARK 5.1.6 (Riemann's principle of removable singularities). If f is analytic in a punctured ball centered at z_0 and there exists $k \in \mathbb{Z}_{\geq 0}$ such that

$$\lim_{z\to z_0}(z-z_0)^{k+1}f(z) \equiv \lim_{z\to z_0}(z-z_0)\overbrace{\big((z-z_0)^k f(z)\big)}^{\substack{\text{analytic in a punctured}\\ \text{ball centered at } z_0}} = 0, \tag{5.1.1}$$

by using the above lemma, we immediately see that there exists an analytic function A, analytic near z_0, such that

$$A(z) = (z-z_0)^k f(z) \quad \text{in a punctured ball centered at } z_0. \tag{5.1.2}$$

If $k = 0$, this implies that z_0 is a removable singularity; if $k > 0$, this implies that z_0 is a pole of f.

DEFINITION 5.1.7. Let f as in (5.1.2). If $k = 0$, then we called such z_0 the pole of order 0 (can be either removable singularity or f is analytic near z_0). If $k > 0$ and $A(z_0) \neq 0$, then we say that the pole z_0 has order k.

REMARK 5.1.8. By using a mathematical induction, one can easily see that (5.1.1) implies that the pole has order *at most* k. Therefore one also can refer the removable singularity as the pole of order 0. This remark generalizes Exercise 4.2.5.

EXAMPLE 5.1.9. Suppose that f has an isolated singularity at $x_0 = 0$ (says) and there exists $C_0 > 0$ such that it satisfies $|f(z)| \leq \frac{C_0}{|z|^\alpha}$ in a punctured ball centered at 0 for some $\alpha > 0$ with $\alpha \notin \mathbb{Z}$. Let $\lceil \alpha \rceil$ be the smallest integer that $\geq \alpha$, and let $\lfloor \alpha \rfloor$ be the largest integer that $\leq \alpha$. One sees that

$$\limsup_{z\to 0}|z^{\lceil\alpha\rceil}f(z)| = \limsup_{z\to 0}|z|^{\lceil\alpha\rceil}|f(z)| \leq \limsup_{z\to 0} C_0|z|^{\lceil\alpha\rceil-\alpha} = 0.$$

Then by Remark 5.1.6, one has

$$z^{\lfloor\alpha\rfloor}f(z) = A(z) \quad \text{in a punctured ball centered at } 0$$

for some analytic function A. Hence it is not possible to find $C_1 > 0$ and $\lfloor\alpha\rfloor < \beta \leq \alpha$ such that $|f(z)| \geq \frac{C_1}{|z|^\beta}$ in a punctured ball centered at 0 (otherwise one can easily obtain a contradiction).

If f has an essential singularity at z_0, then one sees that

$$\text{if } \lim_{z\to z_0}(z-z_0)^{k+1}f(z) \text{ exists for some } k \in \mathbb{Z}_{\geq 0}, \text{ then } \lim_{z\to z_0}(z-z_0)^{k+1}f(z) \neq 0,$$

otherwise we can immediately obtain a contradiction from Remark 5.1.6. In this case, it is not difficult see that $\lim_{z\to z_0}|f(z)| = \infty$. But, however, we do not know whether $\lim_{z\to z_0}(z-z_0)^{k+1}f(z)$ exists or not. We now closing this section by the following theorem.

THEOREM 5.1.10. *If f has an essential singularity at z_0, then for each $R > 0$ the set $f(B_R(z_0)\setminus\{z_0\}) := \{\ f(z) \mid z \in B_R(z_0)\setminus\{z_0\}\ \}$ is dense in $\mathbb{C}$.*

PROOF. Suppose the contrary, that there exists a ball $B_\delta(w_0)$ in $\mathbb{C}$ such that

$$B_\delta(w_0) \cap f(B_R(z_0) \setminus \{z_0\}) = \emptyset.$$

This means that $|f(z) - w_0| \geq \delta$ for all $z \in B_R(z_0) \setminus \{z_0\}$, therefore

$$\left| \frac{1}{f(z) - w_0} \right| \leq \frac{1}{\delta} \quad \text{for all } z \in B_R(z_0) \setminus \{z_0\}.$$

By using Remark 5.1.5, it follows that there exists a function A, which is analytic near z_0, such that

$$\frac{1}{f(z) - w_0} = A(z) \iff f(z) = w_0 + \frac{1}{A(z)}$$

in a punctured ball centered at z_0. This implies that f has either a pole at z_0 (if $A(z_0) = 0$) or a removable singularity at z_0 (if $A(z_0) \neq 0$), which is a singularity. $\square$

5.2. Laurent expansions

We now introduce a powerful tool to help us to study the isolated singularities.

DEFINITION 5.2.1. Let $\{\mu_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$. We say that $\sum_{k\in\mathbb{Z}} \mu_k = L$ for some $L \in \mathbb{C}$ if both $\sum_{k=0}^\infty \mu_k$ and $\sum_{k=-\infty}^{-1} \mu_k \equiv \sum_{k=1}^\infty \mu_{-k}$ converge and satisfies

$$\sum_{k=0}^\infty \mu_k + \sum_{k=-\infty}^{-1} \mu_k = L.$$

We first show that the Laurent expansion make senses:

LEMMA 5.2.2. *The Laurent expansion $f(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ is converge in the domain*

$$A_{R_1,R_2} = \left\{ \; z \in \mathbb{C} \;\middle|\; R_1 < |z| < R_2 \; \right\} \tag{5.2.1}$$

where

$$R_1 = \limsup_{k\to+\infty} |a_{-k}|^{\frac{1}{k}}, \quad R_2 = \left(\limsup_{k\to+\infty} |a_k|^{\frac{1}{k}} \right)^{-1}. \tag{5.2.2}$$

If $0 \leq R_1 < R_2 \leq +\infty$, then f is analytic in the annulus Ω.

PROOF. By using Theorem 2.2.2, one sees that

$$f_1(z) = \sum_{k=0}^\infty a_k z^k \text{ converges and it is an analytic function on } B_{R_2}.$$

If $R_2 = +\infty$, we interpret B_{R_2} as the whole complex plane $\mathbb{C}$. On the other hand, we also see that

$$f_2(z) := \sum_{k=1}^\infty a_{-k} \left(\frac{1}{z}\right)^k \equiv \sum_{k=-\infty}^{-1} a_k z^k \text{ converges for those } z \in \mathbb{C} \text{ with } \frac{1}{|z|} = \left|\frac{1}{z}\right| < \frac{1}{R_1}.$$

In particular,

$$f_2 \text{ converges and it is an analytic function on } \mathbb{C} \setminus \overline{B_{R_1}}.$$

Hence we conclude the theorem with $f = f_1 + f_2$. $\square$

The following theorem shows that the Laurent series will be a very powerful tool to study the singularities.

THEOREM 5.2.3. *If f is analytic in the annulus A_{R_1,R_2} (5.2.1) with $0 \le R_1 < R_2 \le +\infty$, then f has a Laurent expansion $f(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ in A_{R_1,R_2}.*

PROOF. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ represent circles centered at 0 of radii r_1 and r_2 respectively, with $R_1 < r_1 < r_2 < R_2$, with counterclockwise orientation. We fix $z \in B_{r_2} \setminus \overline{B_{r_1}}$ and see that

$$g(w) = \frac{f(w) - f(z)}{w - z}$$

is analytic at $w \in A_{R_1,R_2}$, and by Cauchy closed curve theorem (Theorem 3.3.11), we see that

$$\int_{\mathcal{C}_2 \cup \mathcal{C}_1^{\mathrm{rev}}} g(w)\,\mathrm{d}w = 0,$$

where $\mathcal{C}_1^{\mathrm{rev}}$ is given by Lemma 3.1.8. One has to be careful that the annulus is not simply connected (Example 3.3.2). However, this problem can be overcomed by splitting the annulus as showed in the following diagram:

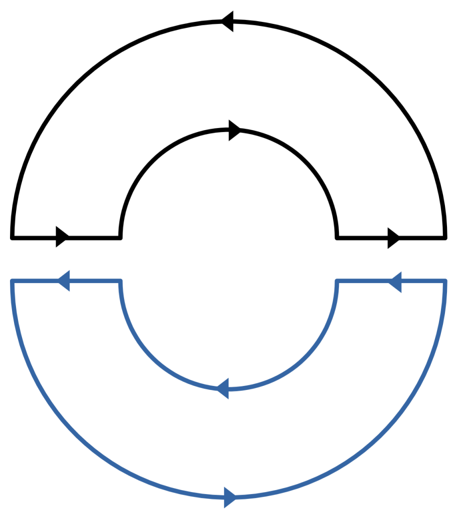


FIGURE 5.2.1. Splitting the contour $\mathcal{C}_2 \cup \mathcal{C}_1^{\mathrm{rev}}$ into two closed curves

Combining the above two equations, we reach

$$\begin{aligned}\int_{\mathcal{C}_2 \cup \mathcal{C}_1^{\mathrm{rev}}} \frac{f(w)}{w-z}\,\mathrm{d}w &= f(z) \int_{\mathcal{C}_2 \cup \mathcal{C}_1^{\mathrm{rev}}} \frac{1}{w-z}\,\mathrm{d}w \\ &= f(z) \left(\overbrace{\int_{\mathcal{C}_2} \frac{1}{w-z}\,\mathrm{d}w}^{=2\pi\mathbf{i}} - \overbrace{\int_{\mathcal{C}_1} \frac{1}{w-z}\,\mathrm{d}w}^{=0} \right) = 2\pi\mathbf{i} f(z) \quad \text{for all } z \in B_{r_2} \setminus \overline{B_{r_1}},\end{aligned}$$

where the first term is due to Cauchy integral formula (Theorem 4.5.1) and the second term is simply by the Cauchy closed curve theorem (Theorem 3.3.11). Hence we reach

$$2\pi\mathbf{i} f(z) = \int_{\mathcal{C}_2} \frac{f(w)}{w-z}\,\mathrm{d}w - \int_{\mathcal{C}_1} \frac{f(w)}{w-z}\,\mathrm{d}w \quad \text{for all } z \in B_{r_2} \setminus \overline{B_{r_1}}.$$

Since $|w| > |z|$ for all $w \in \mathcal{C}_2$, then recall the geometric sequence (see e.g. the proof of Theorem 4.6.1)

$$\frac{1}{w-z} = \frac{1}{w(1-\frac{z}{w})} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}} \quad \text{for all } w \in \mathcal{C}_2,$$

which converges uniformly on $\mathcal{C}_2$. Since $|w| < |z|$ for all $w \in \mathcal{C}_1$, similarly we have the geometric sequence

$$\frac{1}{w-z} = \frac{-1}{z-w} = -\frac{1}{z} - \frac{w}{z^2} - \frac{w^2}{z^3} - \cdots = -\sum_{k=0}^{\infty} \frac{w^k}{z^{k+1}} \quad \text{for all } w \in \mathcal{C}_1,$$

which converges uniformly on $\mathcal{C}_1$. Combining the above three equations, we reach

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_2} \frac{f(w)}{w^{k+1}} \, \mathrm{d}w \right) z^k + \sum_{k=0}^{\infty} \left(\frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_1} f(w) w^k \, \mathrm{d}w \right) z^{-k-1} \\ &= \sum_{k=0}^{\infty} \overbrace{\left(\frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_2} \frac{f(w)}{w^{k+1}} \, \mathrm{d}w \right)}^{= a_k \text{ with } k \geq 0} z^k + \sum_{k=-\infty}^{-1} \overbrace{\left(\frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_1} \frac{f(w)}{w^{k+1}} \, \mathrm{d}w \right)}^{= a_k \text{ with } k < 0} z^k \end{aligned}$$

for all $z \in B_{r_2} \setminus \overline{B_{r_1}}$. Since $\frac{f(w)}{w^{k+1}}$ is analytic on the annulus Ω, by using Cauchy closed curve theorem (Theorem 3.3.11) and the technique sketched by Figure 5.2.1, one sees that for each $k \in \mathbb{Z}$ that

$$a_k = \frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}} \frac{f(w)}{w^{k+1}} \, \mathrm{d}w \tag{5.2.3}$$

for all counterclockwise circle $\mathcal{C}$ centered at 0, hence each a_k is actually independent of r_1 and r_2. Hence we conclude our theorem. ☐

We now state and proof the following representation theorem.

Theorem 5.2.4. *If f is analytic in the annulus $A_{R_1,R_2}(z_0) = \{ z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2 \}$ with $0 \leq R_1 < R_2 \leq \infty$, then f has a unique representation*

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z-z_0)^k, \quad a_k = \frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_R(z_0)} \frac{f(z)}{(z-z_0)^{k+1}} \, \mathrm{d}z \tag{5.2.4}$$

for any counterclockwise circle $\mathcal{C}_R(z_0)$ centered at z_0 with radius R provided $R_1 < R < R_2$.

Proof. It is easy to see that we only need to prove the proposition for $z_0 = 0$. Since $f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$ converges in the annulus A_{R_1,R_2}, then it converges uniformly along $\mathcal{C}$, and thus

$$\int_{\mathcal{C}} \frac{f(z)}{z^{n+1}} \, \mathrm{d}z = \sum_{k \in \mathbb{Z}} a_k \int_{\mathcal{C}} z^{k-n-1} \, \mathrm{d}z \quad \text{for any } n \in \mathbb{Z}. \tag{5.2.5}$$

By using the Cauchy integral formula (Theorem 4.5.1), one has

$$\int_{\mathcal{C}} z^m \, \mathrm{d}z = 0 \quad \text{for all } m \in \mathbb{Z}_{\geq 0}.$$

By using the Cauchy integral formula (Theorem 4.5.1), we have

$$\int_{\mathcal{C}} z^{-1}\,\mathrm{d}z = 2\pi\mathbf{i}.$$

By using the fundamental theorem of line integral (Theorem 3.1.16), one also see that

$$\int_{\mathcal{C}} z^{-m}\,\mathrm{d}z = 0 \quad \text{for all } m \in \mathbb{Z}_{\geq 2}.$$

For future convenience, we record the above three equations as in below:

$$\int_{\mathcal{C}} z^{m}\,\mathrm{d}z = \begin{cases} 2\pi\mathbf{i} & , m = -1, \\ 0 & , m \in \mathbb{Z} \setminus \{-1\}. \end{cases} \tag{5.2.6}$$

Combining (5.2.5) and (5.2.6), we reach

$$\int_{\mathcal{C}} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z = a_n \int_{\mathcal{C}} z^{-1}\,\mathrm{d}z = 2\pi\mathbf{i}a_n \quad \text{for all } n \in \mathbb{Z},$$

which conclude our proposition. □

We now consider the case when z_0 is an isolated singularity. If $R_1 = 0$ and $R_2 < \infty$, then $A_{R_1,R_2} = B_{R_2}(z_0) \setminus \{z_0\}$, i.e. the punctured ball we consider in the previous section. Let f be an analytic function on $B_R(z_0) \setminus \{z_0\}$. By Theorem 5.2.4, f has a unique Laurent series representation

$$f(z) = \sum_{k\in\mathbb{Z}} a_k (z - z_0)^k \quad \text{for all } z \in B_R(z_0) \setminus \{z_0\}. \tag{5.2.7}$$

Definition 5.2.5. We called $\sum_{k\geq 0} a_k(z - z_0)^k$ the *analytic part* of f, while $\sum_{k<0} a_k(z - z_0)^k$ the *principal part* of f.

Since the analytic part of f does nothing with the singularity, we are now interested in the principal part of f. From (5.2.7) we now able to give a full characterization for isolated singularities in terms of Laurent series:

Theorem 5.2.6. *Let f be an analytic function on a punctured ball centered at z_0. By Theorem 5.2.4, f has a unique Laurent series representation (5.2.7). Then either one of the following must holds:*

- (i) *If f has a pole at z_0 of order 0 (i.e. removable singularity or f is analytic near z_0), then $C_{-k} = 0$ for all $k \in \mathbb{N}$.*
- (ii) *If f has a pole at z_0 of order $n \in \mathbb{N}$, then $C_{-n} \neq 0$ and $C_{-k} = 0$ for all $k > n$. In other words, the principal part of f is simply $\mathcal{P}\left(\frac{1}{z-z_0}\right)$ for some polynomial $\mathcal{P}$ with degree n.*
- (iii) *If f has an essential singularity at z_0, then $C_{-k} \neq 0$ for infinitely many $k \in \mathbb{N}$.*

Remark. Let g be an analytic function in an open set Ω. Suppose that $z_0 \in \Omega$ is a zero of g, then we consider its power series around z_0 (Theorem 4.6.1):

$$g(z) = \sum_{k=0}^{\infty} C_k (z - z_0)^k.$$

From $g(z_0) = 0$, one has $C_0 = 0$. If g is nontrivial, then there exists $k_0 \in \mathbb{N}$ such that $C_{k_0} \neq 0$ and $C_k = 0$ for all $0 \leq k < k_0$, and we write

$$g(z) = \sum_{k=k_0}^{\infty} C_k (z - z_0)^k = (z - z_0)^{k_0} \left(\sum_{\ell=0}^{\infty} C_{\ell+k_0} (z - z_0)^{\ell} \right),$$

which means that the zero z_0 must have finite order. This also implies that each pole must have finite order, therefore all isolated singularities are actually classified by Theorem 5.2.6.

Proof of (i). By definition, there exists a function A, analytic near z_0, such that $f(z) = A(z)$ in a punctured ball centered at z_0. Then by Theorem 5.2.4, the Laurent series of f must equal to the power series of A. □

Proof of (ii). By definition, one writes

$$f(z) = \frac{A(z)}{(z - z_0)^n} \quad \text{in a punctured ball centered at } z_0,$$

where A is analytic near z_0. Using the local power series representation (Theorem 4.6.1), we write $A(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ and we see that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k-n} = \sum_{j=-n}^{\infty} a_{n+j} (z - z_0)^j$$

in a punctured ball centered at z_0. Finally, by Theorem 5.2.4, the above equation representations the unique Laurent series of f, which conclude our theorem. □

Proof of (iii). Suppose the contrary, there exists $n \in \mathbb{N}$ such that $C_{-k} = 0$ for all $k > n$. Riemann's principle of removable singularities (Remark 5.1.6) shows that z_0 is pole, which is a contradiction. □

Finally, we closed this section by exhibit an application of the representation formula of Laurent series – together with Liouville theorem and fundamental theorem of algebra – in abstract algebra (field theory).

Theorem 5.2.7 (Partial fraction decomposition of rational functions). *Any proper rational function $\frac{P(z)}{Q(z)}$, where P and Q are polynomials with $\deg P < \deg Q$, can be expanded as a sum of polynomials in $\frac{1}{z-z_k}$, where $\{z_1, z_2, \cdots, z_n\}$ are the set of distinct zeros of Q.*

Sketch of proof. By using fundamental theorem of algebra (Theorem 4.3.6), we can write $Q(z) = A(z - z_1)^{k_1}(z - z_2)^{k_2} \cdots (z - z_n)^{k_n}$ for some $n \leq \deg Q$. This shows that $\frac{P(z)}{Q(z)}$ has a pole of order at most k_j at z_j.

(1) Using Theorem 5.2.6, the principal part of $A_0(z) := \frac{P(z)}{Q(z)}$ near z_1 takes the form $\mathcal{P}_1\left(\frac{1}{z-z_1}\right)$ polynomial $\mathcal{P}_1$. Clearly, $\mathcal{P}_1\left(\frac{1}{z-z_1}\right)$ is analytic in $\mathbb{C} \setminus \{z_1\}$. We now define $A_1(z) := \frac{P(z)}{Q(z)} - \mathcal{P}_1\left(\frac{1}{z-z_1}\right)$.

(2) Using Theorem 5.2.6, the principal part of $A_1(z)$ near z_2, takes the form $\mathcal{P}_2\left(\frac{1}{z-z_2}\right)$ polynomial $\mathcal{P}_2$. Clearly, $\mathcal{P}_2\left(\frac{1}{z-z_2}\right)$ is analytic in $\mathbb{C} \setminus \{z_2\}$. We now define $A_2(z) := \frac{P(z)}{Q(z)} - \mathcal{P}_1\left(\frac{1}{z-z_1}\right) - \mathcal{P}_2\left(\frac{1}{z-z_2}\right)$.

By repeting the above steps (can be rigorously written down using mathematical induction), one sees that

$$\frac{P(z)}{Q(z)} - \mathcal{P}_1\left(\frac{1}{z-z_1}\right) - \cdots - \mathcal{P}_n\left(\frac{1}{z-z_n}\right)$$

is an entire function. Since $\deg P < \deg Q$, by taking $|z| \to \infty$, we see that actually above entire function is bounded. Therefore the Liouville theorem (Theorem 4.3.1) implies that there exists a constant $C \in \mathbb{C}$ such that

$$\frac{P(z)}{Q(z)} - \mathcal{P}_1\left(\frac{1}{z-z_1}\right) - \cdots - \mathcal{P}_n\left(\frac{1}{z-z_n}\right) \equiv C \quad \text{for all } z \in \mathbb{C},$$

which conclude our theorem[1]. □

5.3. Winding numbers and the Cauchy residue theorem

Let f be an analytic function on a punctured ball centered at z_0. By using Theorem 5.2.4, one can write

$$f(z) = \sum_{k\in\mathbb{Z}} a_k(z-z_0)^k, \quad a_k = \frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z \tag{5.3.1}$$

for any counterclockwise circle $\mathcal{C}$ centered at z_0 (within the analyticity region of f). From (5.2.6), we reach

$$\int_{\mathcal{C}} f = 2\pi\mathbf{i}a_{-1}.$$

This suggests the coefficient a_{-1} is of special significance in this context.

Definition 5.3.1. The coefficient a_{-1} is called the *residue* of f at z_0, and we denote $\mathrm{Res}\,(f; z_0) := a_{-1}$.

Proposition 5.3.2 (Evaluation of residues via complex differentiation). *If f has a pole of order $k \in \mathbb{N}$ at z_0, then*

$$\mathrm{Res}\,(f; z_0) = \frac{1}{(k-1)!}\,\partial_z^{k-1}\left((z-z_0)^k f(z)\right)\big|_{z\to z_0}.$$

Remark. Intuitively, we want to remove the pole of f by multiplying $(z-z_0)^k$. The "price" of doing so is some complex differentiations.

Proof. By Theorem 5.2.6, one can write

$$f(z) = a_{-k}(z-z_0)^{-k} + \cdots + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \cdots.$$

Then we see that

$$(z-z_0)^k f(z) = a_{-k} + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + a_1(z-z_0)^{k+1} + \cdots,$$

and hence

$$\partial_z^{k-1}\left((z-z_0)^k f(z)\right) = (k-1)!a_{-1} + a_0 k!(z-z_0) + \cdots.$$

Evaluate $z = z_0$ in the above equation, we conclude our proposition. □

Remark 5.3.3. In most cases of higher-order poles, as with essential singularities, the most convenient way to determine the residue is directly from the Laurent expansion.

[1] In fact, since $\deg P < \deg Q$, by taking $|z| \to \infty$, we see that indeed $C = 0$.

To evaluate $\int_\gamma f$ when γ is a general closed curve (and when f may have isolated singularities), we introduce the following concept.

DEFINITION 5.3.4. Suppose that γ is a parametrizable continuous piecewise-C^1 closed curve and that $a \notin \gamma$. Then the number

$$\operatorname{wind}(\gamma, a) = \frac{1}{2\pi\mathbf{i}} \int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$

is called the *winding number* of γ around a.

If $\gamma = \mathcal{C}$ be the counterclockwise circle $\mathcal{C}$, then by Cauchy closed curve theorem (Theorem 3.3.11) we see that

$$\operatorname{wind}(\gamma, a) = \begin{cases} 1 & \text{if } a \text{ is inside the circle,} \\ 0 & \text{if } a \text{ is outside the circle.} \end{cases}$$

If γ circles the point a k-times via the parametrization $\gamma = \left[z_0 + re^{\mathbf{i}\theta} : 0 \le \theta \le 2k\pi\right]$, then

$$\operatorname{wind}(\gamma, a) = \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \mathbf{i}\,\mathrm{d}\theta = k,$$

which suggests the terminology "winding number". We now need to prove this idea make senses for general closed curve.

For each fixed parametrizable continuous piecewise-C^1 closed curve γ, it is important to observe that

$$\text{the mapping } a \mapsto \operatorname{wind}(\gamma, a), \text{ also can be denoted by } \operatorname{wind}(\gamma, \cdot),$$

is continuous as long as $a \notin \gamma$.

PROPOSITION 5.3.5. *For any parametrizable continuous piecewise-C^1 closed curve γ and $a \notin \gamma$, the winding number* $\operatorname{wind}(\gamma, a)$ *is an integer. In addition, the mapping* $\operatorname{wind}(\gamma, \cdot)$ *is locally constant (i.e. it is constant in the connected open components of $\mathbb{C} \setminus \gamma$).*

PROOF. Write $\gamma = \left[\ z(t) \mid 0 \le t \le 1\ \right]$, and set

$$F(s) = \int_0^s \frac{\dot{z}(t)}{z(t) - a}\,\mathrm{d}t \quad \text{for } 0 \le s \le 1,$$

where $\dot{z}$ denotes the differentiation of z with respect to t (see Definition 3.1.2). By fundamental theorem of calculus on $\mathbb{R}$, one sees that

$$\dot{F}(s) = \frac{\dot{z}(s)}{z(s) - a} \quad \text{for all } 0 < s < 1,$$

and thus (by the technique of integral factor, should be taughted in ODE course)

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((z(s) - a)e^{-F(s)}\right) = 0 \quad \text{for all } 0 < s < 1.$$

Since the open interval $(0,1)$ is connected, then

$$(z(s) - a)e^{-F(s)} \equiv C \quad \text{for all } 0 \le s \le 1$$

for some constant $C \in \mathbb{C}$. Note: the equation also holds for endpoints $s = 0$ and $s = 1$, because F and z are continuous on $[0,1]$. Therefore, we have

$$(z(s) - a)e^{-F(s)} = z(0) - a \quad \text{for all } 0 \le s \le 1.$$

Since $a \notin \gamma$, then $z(0) - a \neq 0$, and then we have

$$e^{F(s)} = \frac{z(s) - a}{z(0) - a} \quad \text{for all } 0 \leq s \leq 1.$$

Since γ is a closed curve, then $z(1) = z(0)$, and then

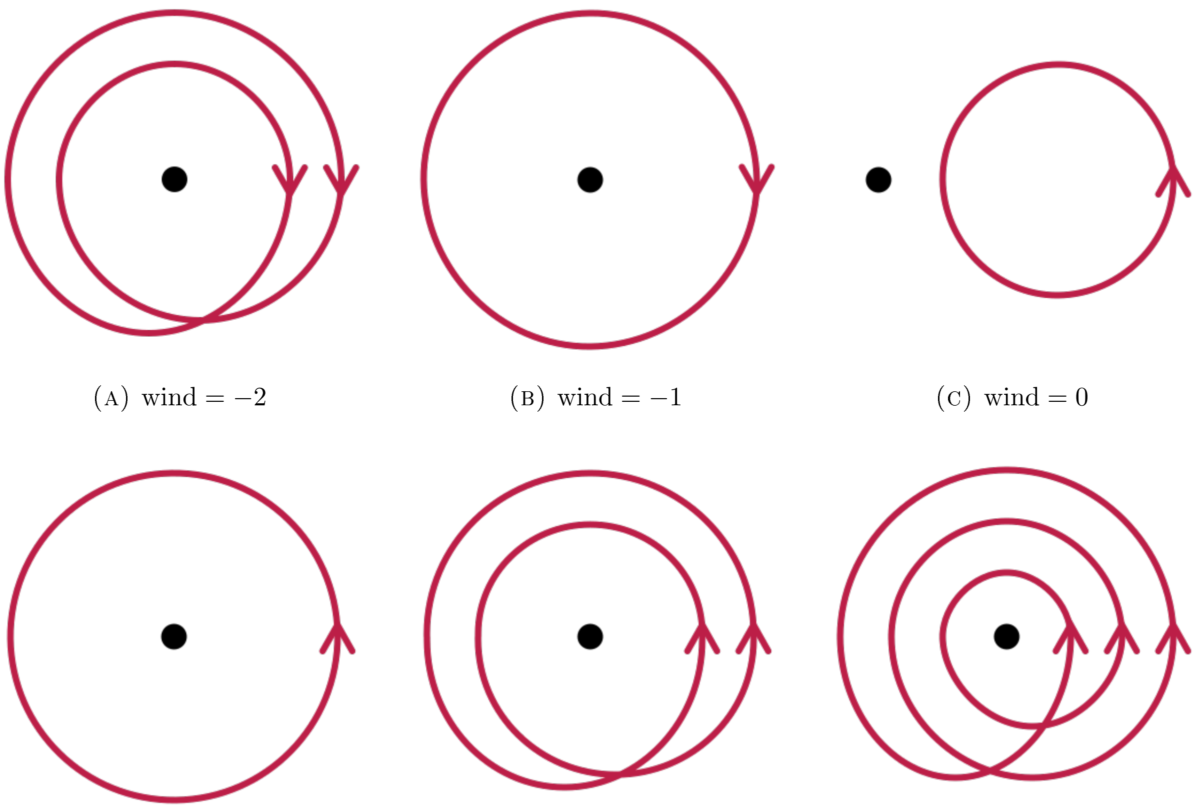
$$e^{F(1)} = \frac{z(1) - a}{z(0) - a} = 1.$$

This implies that

$$F(1) = 2\pi \mathbf{i} k \quad \text{for some integer } k \in \mathbb{Z},$$

and hence we conclude that $\operatorname{wind}(\gamma, a) = \frac{1}{2\pi \mathbf{i}} F(1) = k$. $\square$

Here we exhibit some graphical examples from Wikipedia:



(A) wind $= -2$ (B) wind $= -1$ (C) wind $= 0$

(D) wind $= 1$ (E) wind $= 2$ (F) wind $= 3$

FIGURE 5.3.1. Winding numbers (By Jim.belk - Own work, Public Domain)

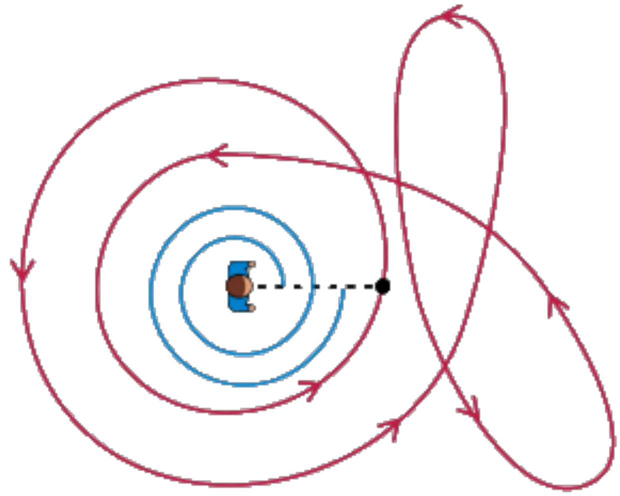


FIGURE 5.3.2. wind $(\gamma, a) = 2$ (By Jim.belk - Own work, Public Domain)

We finally able to prove the following theorem.

THEOREM 5.3.6 (Cauchy residue theorem). *Suppose f is analytic in a simply connected open set Ω except for isolated singularities at $z_1, z_2, \cdots, z_m \in \Omega$. Let γ be a parametrizable continuous piecewise-C^1 closed curve in Ω, which not intersecting any of the singularities. Then*

$$\int_\gamma f = 2\pi \mathbf{i} \sum_{k=1}^{m} \mathrm{wind}\,(\gamma, z_k)\,\mathrm{Res}\,(f; z_k).$$

REMARK (Cauchy closed curve theorem). For those f which is analytic in a simply connected open set Ω, one has $\mathrm{Res}\,(f; z) = 0$ for all $z \in \Omega$, which can be easily see from Definition 5.3.1. Therefore one has $\int_\gamma f = 0$. Therefore the Cauchy closed curve theorem (Theorem 3.3.11) is a special case of Cauchy residue theorem above.

REMARK (Cauchy integral formula). By considering $f(z) = \frac{g(z)}{z-a}$ with analytic function g and $a \in \Omega$, one has $\mathrm{Res}\,(f; a) = g(a)$, which can be easily see from Definition 5.3.1. If we choose $\mathcal{C}$ be a parametrizable continuous piecewise-C^1 closed curve in Ω, which not intersecting a and is simple (i.e. wind $(\gamma, a) = 1$), one sees that

$$\int_{\mathcal{C}} \frac{g(z)}{z-a}\,\mathrm{d}z = 2\pi \mathbf{i}\,\mathrm{Res}\,(f; a) = g(a).$$

Therefore the Cauchy integral formula (Theorem 4.5.1) is a special case of Cauchy residue theorem above.

PROOF OF THEOREM 5.3.6. Similar to Theorem 5.2.7, if we subtract the principal parts

$$\mathcal{P}_1\left(\frac{1}{z-z_1}\right), \cdots, \mathcal{P}_m\left(\frac{1}{z-z_m}\right)$$

from f, one sees that the difference

$$g(z) = f(z) - \sum_{k=1}^{m} \mathcal{P}_k\left(\frac{1}{z-z_k}\right)$$

is analytic on D. Hence the Cauchy closed curve theorem (Theorem 3.3.11) implies that

$$0 = \int_\gamma g = \int_\gamma f - \sum_{k=1}^m \int_\gamma \mathcal{P}_k\left(\frac{1}{z-z_k}\right). \tag{5.3.2}$$

By the definition of principal part (Definition 5.2.5) and the definition of residual (Definition 5.3.1), one sees that

$$\mathcal{P}_k\left(\frac{1}{z-z_k}\right) = \frac{\mathrm{Res}\,(f,z_k)}{z-z_k} + \frac{a_{-2}}{(z-z_k)^2} + \frac{a_{-3}}{(z-z_k)^3} + \cdots,$$

and the above sequence converges uniformly on γ. By using the fundamental theorem of line integral (Theorem 3.1.16), it is easy to see that

$$\int_\gamma \frac{1}{(z-z_k)^k}\,\mathrm{d}z = 0 \quad \text{for all } k = 2,3,4,\cdots,$$

because γ is a closed curve. Hence we see that

$$\int_\gamma \mathcal{P}_k\left(\frac{1}{z-z_k}\right) = \mathrm{Res}\,(f,z_k)\int_\gamma \frac{1}{z-z_k}\,\mathrm{d}z = 2\pi\mathbf{i}\,\mathrm{wind}\,(\gamma,z_k)\,\mathrm{Res}\,(f;z_k).$$

Plugging the above equation into (5.3.2), we conclude our theorem. □

5.4. Some applications in combinatorics : Egorychev method

The connection between binomial coefficients and contour integration is an immediate corollary of the Residue theorem (Theorem 5.3.6). These techniques sometimes also referred as the *Egorychev method*, which is a collection of techniques introduced by Georgy Egorychev for finding identities among sums of binomial coefficients, Stirling numbers, Bernoulli numbers, Harmonic numbers, Catalan numbers and other combinatorial numbers [**Ego84**].

THEOREM 5.4.1 (First binomial coefficient integral). *For each $n \in \mathbb{N}$ and $k = 0,1,\cdots,n$, one has*

$$\binom{n}{k} = \frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}} \frac{(1+z)^n}{z^{k+1}}\,\mathrm{d}z \tag{5.4.1}$$

for all simple closed (parametrizable continuous piecewise-C^1) curve $\mathcal{C}$ surrounding the origin.

PROOF. For each $k = 0,1,\cdots,n$, by choosng

$$f(z) = \frac{(1+z)^n}{z^{k+1}} = \sum_{j=0}^n \binom{n}{j} z^{j-k-1}$$

in the Residue theorem (Theorem 5.3.6), one sees that

$$\begin{aligned}\int_{\mathcal{C}} \frac{(1+z)^n}{z^{k+1}}\,\mathrm{d}z &= 2\pi\mathbf{i}\,\mathrm{Res}\,(f;0) \quad (\text{Theorem 5.3.6}) \\ &= 2\pi\mathbf{i}\binom{n}{k} \quad (\text{Definition 5.3.1})\end{aligned}$$

where we interpret $n(n-1)\cdots(n-k+1) = 1$ when $k = 0$, which conclude the following theorem. □

EXAMPLE 5.4.2. Let $\mathcal{C}$ be any simple closed (parametrizable continuous piecewise-C^1) curve surrounding the origin. By using (5.4.1), it is easy to see that

$$\begin{aligned}
&\binom{n-1}{k-1} + \binom{n-1}{k} \\
&\quad = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{(1+z)^{n-1}}{z^k} \,\mathrm{d}z + \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{(1+z)^{n-1}}{z^{k+1}} \,\mathrm{d}z \\
&\quad = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{(1+z)^{n-1}z + (1+z)^{n-1}}{z^{k+1}} \,\mathrm{d}z \\
&\quad = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{(1+z)^{n}}{z^{k+1}} \,\mathrm{d}z = \binom{n}{k},
\end{aligned}$$

which is the well-known *Pascal triangle*.

EXAMPLE 5.4.3 (A special case of Chu-Vandermonde identity). Let $\mathcal{C}$ be any simple closed (parametrizable continuous piecewise-C^1) curve surrounding the origin. By using binomial theorem, one sees that

$$(1+z)^n(1+z^{-1})^n = \sum_{k=0}^{n} \binom{n}{k} z^k \cdot \sum_{\ell=0}^{n} \binom{n}{\ell} z^{-\ell} = \sum_{k=0}^{n}\sum_{\ell=0}^{n} \binom{n}{k}\binom{n}{\ell} z^{k-\ell}.$$

By choosing $f(z) = \frac{(1+z)^n(1+z^{-1})^n}{z}$, one sees that

$$\begin{aligned}
\int_{\mathcal{C}} \frac{(1+z)^n(1+z^{-1})^n}{z} \,\mathrm{d}z &= 2\pi\mathbf{i} \operatorname{Res}(f;0) \quad \text{(Theorem 5.3.6)} \\
&= 2\pi\mathbf{i} \sum_{k=0}^{n} \binom{n}{k}^2 \quad \text{(Definition 5.3.1)}.
\end{aligned}$$

On the other hand, we compute that

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k}^2 &= \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{(1+z)^n(1+z^{-1})^n}{z} \,\mathrm{d}z \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{(1+z)^n(z+1)^n}{z^{n+1}} \,\mathrm{d}z \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{(1+z)^{2n}}{z^{n+1}} \,\mathrm{d}z = \binom{2n}{n},
\end{aligned}$$

where the last equality is given by (5.4.1).

EXAMPLE 5.4.4. We now want to prove the binomial identity:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\binom{k}{j} = (-1)^n \binom{n}{j}\binom{n+j}{n}.$$

By using the first binomial coefficient integral (Theorem 5.4.1), one has

$$\binom{n+k}{k} = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}_r} \frac{(1+z)^{n+k}}{z^{k+1}} \,\mathrm{d}z, \quad \binom{k}{j} = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}_s} \frac{(1+w)^{k}}{w^{j+1}} \,\mathrm{d}w$$

for some $r > 0$ and $s > 0$, where $\mathcal{C}_\rho$ is the circle with radius ρ centered at 0, which is counterclockwise oriented. This yields

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\binom{k}{j} \\
&\quad = \frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^n}{z}\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_s}\frac{1}{w^{j+1}}\overbrace{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{(1+z)(1+w)}{z}\right)^k}^{\text{binomial theorem}}\,\mathrm{d}w\,\mathrm{d}z \\
&\quad = \frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^n}{z}\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_s}\frac{1}{w^{j+1}}\left(1-\frac{(1+z)(1+w)}{z}\right)^n\mathrm{d}w\,\mathrm{d}z \\
&\quad = \frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_s}\frac{1}{w^{j+1}}\left(z-(1+z)(1+w)\right)^n\mathrm{d}w\,\mathrm{d}z \\
&\quad = \frac{(-1)^n}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_s}\frac{1}{w^{j+1}}\overbrace{(1+w(1+z))^n}^{\text{binomial theorem}}\,\mathrm{d}w\,\mathrm{d}z \\
&\quad = \frac{(-1)^n}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_s}\frac{1}{w^{j+1}}\sum_{q=0}^{n}\binom{n}{q}w^q(1+z)^q\,\mathrm{d}w\,\mathrm{d}z
\end{aligned}$$

$$= \frac{(-1)^n}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_s}\sum_{q=0}^{n}\binom{n}{q}w^{q-j-1}(1+z)^q\,\mathrm{d}w\,\mathrm{d}z. \tag{5.4.2}$$

We now define

$$f(w) := \sum_{q=0}^{n}\binom{n}{q}w^{q-j-1}(1+z)^q.$$

By Definition 5.3.1, it is easy to see that $\operatorname{Res}(f;0) = \binom{n}{j}(1+z)^j$, and thus by using the Residue theorem (Theorem 5.3.6), one sees that

$$\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_s}\sum_{q=0}^{n}\binom{n}{q}w^{q-j-1}(1+z)^q\,\mathrm{d}w = \binom{n}{j}(1+z)^j.$$

Plugging the above equation into (5.4.2), we reach

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{k}{j} \\
&\quad = \frac{(-1)^n}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^n}{z^{n+1}}\binom{n}{j}(1+z)^j\,\mathrm{d}z \\
&\quad = \binom{n}{j}\frac{(-1)^n}{2\pi\mathbf{i}}\int_{\mathcal{C}_r}\frac{(1+z)^{n+j}}{z^{n+1}}\,\mathrm{d}z \\
&\quad = (-1)^n\binom{n}{j}\binom{n+j}{n},
\end{aligned}$$

where the last identity follows from the first binomial coefficient integral (Theorem 5.4.1).

THEOREM 5.4.5 (Second binomial coefficient integral). *For each $n \in \mathbb{N}$ and $k = 0, 1, \cdots, n$, one has*

$$\binom{n}{k} = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}_\rho} \frac{1}{(1-z)^{k+1} z^{n-k+1}} \, \mathrm{d}z \tag{5.4.3}$$

for all $0 < \rho < 1$, where $\mathcal{C}_\rho$ is the circle with radius ρ centered at 0, which is counterclockwise oriented.

REMARK. The reason we restrict $0 < \rho < 1$ is to make sure that $\frac{1}{(1-z)^{k+1}}$ is well-defined (as a uniformly converge geometric sequence).

PROOF. For each $k = 0, 1, \cdots, n$, and let

$$f(z) = \frac{1}{(1-z)^{k+1} z^{n-k+1}}.$$

Since f has pole of order $n - k + 1$ at $z_0 = 0$, by using Proposition 5.3.2, one has

$$\begin{aligned}
\mathrm{Res}\,(f;0) &= \frac{1}{(n-k)!} \, \partial_z^{n-k} \left(z^{n-k+1} f(z) \right) \Big|_{z\to 0} \\
&= \frac{1}{(n-k)!} \, \partial_z^{n-k} \left((1-z)^{-k-1} \right) \Big|_{z\to 0} \\
&= \frac{1}{(n-k)!} (k+1) \, \partial_z^{n-k-1} \left((1-z)^{-k-2} \right) \Big|_{z\to 0} \\
&= \frac{1}{(n-k)!} (k+1)(k+2) \, \partial_z^{n-k-2} \left((1-z)^{-k-3} \right) \Big|_{z\to 0} \\
&\;\;\vdots \\
&= \frac{1}{(n-k)!} (k+1)(k+2) \cdots n \\
&= \binom{n}{k}.
\end{aligned}$$

Therefore, by using the Residue theorem (Theorem 5.3.6), we immediately conclude (5.4.3). ☐

EXERCISE 5.4.6. Prove Theorem 5.4.1 by using Residue theorem (Theorem 5.3.6) and evaluation the residues via complex differentiation (Proposition 5.3.2).

THEOREM 5.4.7 (Exponential integral). *For each $n \in \mathbb{N}$ and $k = 0, 1, \cdots, n$, one has*

$$n^k = \frac{k!}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{e^{nz}}{z^{k+1}} \, \mathrm{d}z$$

for all simple closed (parametrizable continuous piecewise-C^1) curve $\mathcal{C}$ surrounding the origin.

EXERCISE 5.4.8. Prove Theorem 5.4.7 by using the Residue theorem (Theorem 5.3.6) [Hint: Consider the function $\frac{e^{nz}}{z^{k+1}}$] .

THEOREM 5.4.9. *For each $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$, one has*

$$\chi_{\{(n,k)\in\mathbb{Z}\times\mathbb{Z}\,:\,n\geq k\}}(n,k) = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}_\rho} \frac{1}{z^{n-k+1}} \frac{1}{1-z} \, \mathrm{d}z$$

for all $0 < \rho < 1$, *where* $\mathcal{C}_\rho$ *is the circle with radius* ρ *centered at 0, which is counterclockwise oriented. Here* χ_A *is the indicator function defined by*

$$\chi_A(x) = \begin{cases} 1 & , x \in A, \\ 0 & , x \notin A. \end{cases}$$

REMARK. The reason we restrict $0 < \rho < 1$ is to make sure that $\frac{1}{1-z}$ is well-defined (as a uniformly converge geometric sequence).

REMARK (Iverson bracket). In many cases, we simplyfied the notations by simply writing $\{n \geq k\} = \{(n,k) \in \mathbb{Z} \times \mathbb{Z} : n \geq k\}$. The Iverson bracket $[\![\cdot]\!]$, given by $[\![x \in A]\!] := \chi_A(x)$. One note that the Kronecker delta can be expressed as $\delta_{ij} = [\![\{i = j\}]\!] \equiv [\![i = j]\!]$. By slightly abusing notations, sometimes we write Theorem 5.4.9 as

$$[\![n \geq k]\!] \equiv \chi_{\{n \geq k\}} = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{C}_\rho} \frac{1}{z^{n-k+1}} \frac{1}{1-z}\,\mathrm{d}z.$$

PROOF OF THEOREM 5.4.9. We consider the function

$$f(z) = \frac{1}{z^{n-k+1}} \frac{1}{1-z}.$$

When $n + 1 \leq k$ (iff $n < k$), then $f(z)$ is analytic in B_1, so $\operatorname{Res}(f;0) = 0$. Otherwise when $n + 1 > k$ (iff $n \geq k$), then $f(z)$ has a pole of order $n - k + 1$ at $z_0 = 0$, and hence by Proposition 5.3.2 we see that

$$\begin{aligned}
\operatorname{Res}(f;0) &= \frac{1}{(n-k)!}\, \partial_z^{n-k} \left(z^{n-k+1} f(z) \right)\Big|_{z \to 0} \\
&= \frac{1}{(n-k)!}\, \partial_z^{n-k} \left((1-z)^{-1} \right)\Big|_{z \to 0} \\
&= \frac{1}{(n-k)!}\, \partial_z^{n-k-1} \left((1-z)^{-2} \right)\Big|_{z \to 0} \\
&= \frac{1}{(n-k)!} 2\, \partial_z^{n-k-2} \left((1-z)^{-3} \right)\Big|_{z \to 0} \\
&= \frac{1}{(n-k)!} 2 \cdot 3\, \partial_z^{n-k-3} \left((1-z)^{-4} \right)\Big|_{z \to 0} \\
&\;\;\vdots \\
&= \frac{1}{(n-k)!} 2 \cdot 3 \cdot \cdots \cdot (n-k) = 1.
\end{aligned}$$

Therefore, by using the Residue theorem (Theorem 5.3.6), we immediately conclude our theorem. □

The *Stirling set number* (also known as the *Stirling number of second kind*) $\left\{ {n \atop k} \right\}$ is the number of ways of partitioning a set of n elements into k nonempty sets, which is given by (`https://dlmf.nist.gov/26.8`)

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

THEOREM 5.4.10. *For each $n \in \mathbb{N}$ and $k = 1, \cdots, n$, one has*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{n!}{k!} \frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}} \frac{(e^z - 1)^k}{z^{n+1}} \, \mathrm{d}z$$

for all simple closed (parametrizable continuous piecewise-C^1) curve $\mathcal{C}$ surrounding the origin.

PROOF. It is easy to see that the function

$$f(z) = \frac{(e^z - 1)^k}{z^{n+1}} = \frac{1}{z^{n+1}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{jz}$$

has a pole of order at most $n+1$ at $z_0 = 0$, and hence by Proposition 5.3.2 we see that

$$\begin{aligned} \mathrm{Res}\,(f;0) &= \frac{1}{n!} \, \partial_z^n \left(z^{n+1} f(z) \right) \big|_{z \to 0} \\ &= \frac{1}{n!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \, \partial_z^n \left(e^{jz} \right) \big|_{z \to 0} \\ &= \frac{1}{n!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \, j^n \\ &= \frac{k!}{n!} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}. \end{aligned}$$

Therefore, by using the Residue theorem (Theorem 5.3.6), we immediately conclude our theorem. ∎

CHAPTER 6

Some special analytic functions

6.1. The analytic function $\log z$

In real analysis, the (natural) logarithmic function $\log x$ for $x > 0$ is defined by the inverse function of the exponential function e^x. The main difficulty to extend this to complex number is the function e^z is not injective.

DEFINITION 6.1.1. We say that f is an *analytic branch* of $\log z$ in a domain D if f is analytic in D and $e^{f(z)} = z$.

REMARK 6.1.2. If f is an analytic branch of $\log z$, then **all** other branches are $g(z) = f(z) + 2\pi k\mathbf{i}$ for $k \in \mathbb{Z}$.

For each $x > 0$, it is well-known that
$$\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}.$$
If we fix any $x_0 > 0$, then the fundamental theorem of calculus implies
$$\log x = \int_{x_0}^{x} \frac{1}{y}\,\mathrm{d}y + \log x_0.$$
This suggests us to define the complex logarithmic as in the following:

THEOREM 6.1.3. *Suppose that D is simply connected and that $0 \notin D$ (this condition is quite natural since $\log 0$ is not well-defined). Choose $z_0 \in D$, fix a value of $\log z_0 \in \mathbb{C}$ such that $e^{\log z_0} = z_0$ and set*
$$f(z) := \int_{z_0}^{z} \frac{1}{\zeta}\,\mathrm{d}\zeta + \log z_0.$$
Then f is an analytic branch of $\log z$ in D, satisfying $f'(z) = \frac{1}{z}$ for all $z \in D$.

REMARK 6.1.4. Here $\int_{z_0}^{z} \frac{1}{\zeta}\,\mathrm{d}\zeta$ means the integral along any paths from z_0 to z. Since $\frac{1}{\zeta}$ is analytic in D, by using the Cauchy residual theorem (Theorem 5.3.6), one sees that $\int_{z_0}^{z} \frac{1}{\zeta}\,\mathrm{d}\zeta$ is indeed independent of the chosen path.

PROOF OF THEOREM 6.1.3. It is easy to see that f is analytic in D with $f'(z) = \frac{1}{z}$. The remaining task is to show $e^{f(z)} = z$. We define
$$g(z) = ze^{-f(z)}.$$
Since $g'(z) = e^{-f(z)} - zf'(z)e^{-f(z)} = 0$ in D and D is simply connected, by using the fundamental theorem of line integral (Theorem 3.1.16) one sees that g is a constant function and
$$g(z) = g(z_0) = z_0 e^{-\log z_0} = 1,$$
hence we conclude $e^{f(z)} = z$. $\square$

In a typical situation (unless stated), we choose $D = \mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}$ and $z_0 = 1$:

DEFINITION 6.1.5. The function $\mathrm{Log}\, z := \int_1^z \frac{1}{\zeta}\,\mathrm{d}\zeta$ for all $z \in \mathbb{C}\setminus\{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}$, which defined in the sense of Remark 6.1.4, is called the (standard) *principal branch* of $\log z$.

It is easy to see that

$$(\mathrm{Log}\, z)' = \frac{1}{z} \quad \text{and} \quad -\pi < \mathfrak{Im}\,(\mathrm{Log}\, z) < \pi.$$

One can use Remark 6.1.2 to construct **all** other branches

$$\mathrm{Log}\, z + 2\pi k\mathbf{i} \quad \text{for all } k \in \mathbb{Z}, \tag{6.1.1}$$

which also corresponding to $D = \mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}$ and $z_0 = 1$ as well. We also can define the logarithms to other bases by

$$\mathrm{Log}_w\, z := \frac{\mathrm{Log}\, z}{\mathrm{Log}\, w}.$$

Recall that $\exp(\mathrm{Log}\, z) = z$ for all $z \in \mathbb{C}\setminus\{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}$, that is, Log is the right-inverse of exp (with respect to the composition operator of functions).

QUESTION 6.1.6. *How about* $\mathrm{Log}\,(\exp w)$ *for* $w \in \mathbb{C}$ *satisfies* $\exp w \in \mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}$*?*

The above question can be easily answered by the following theorem gives an equivalent definition of $\mathrm{Log}\, z$:

THEOREM 6.1.7 (Equivalent definition of principal branch of $\log z$). *For each* $z \in \mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}$*, one can write* $z = Re^{\mathbf{i}\theta}$ *for some* $R > 0$ *and* $-\pi < \theta < \pi$*. Then*

$$\mathrm{Log}\, z = \log R + \mathbf{i}\theta \equiv \log |z| + \mathbf{i}\theta.$$

REMARK 6.1.8 (Left inverse of exponential). For each $w \in \mathbb{C}$, one sees that

$$\exp w = e^{\mathfrak{Re}\, w + \mathbf{i}\mathfrak{Im}\, w} = e^{\mathfrak{Re}\, w} e^{\mathbf{i}\mathfrak{Im}\, w} = e^{\mathfrak{Re}\, w}(\cos(\mathfrak{Im}\, w) + \mathbf{i}\sin(\mathfrak{Im}\, w)).$$

If $-\frac{\pi}{2} < \mathfrak{Im}\, w < \frac{\pi}{2}$, then $\mathfrak{Re}\,(\exp w) = e^{\mathfrak{Re}\, w}\cos(\mathfrak{Im}\, w) > 0$. Therefore, at least when $-\frac{\pi}{2} < \mathfrak{Im}\, w < \frac{\pi}{2}$, one can choose $z = \exp w$ in Theorem 6.1.7 to see that

$$\mathrm{Log}\,(\exp w) = \log(e^{\mathfrak{Re}\, w}) + \mathbf{i}\mathfrak{Im}\, w = \mathfrak{Re}\, w + \mathbf{i}\mathfrak{Im}\, w = w.$$

This shows that the **principal branch** of complex logarithmic is the left inverse of complex exponential in suitable domain, it is valid when $-\frac{\pi}{2} < \mathfrak{Im}\, w < \frac{\pi}{2}$, but not all $w \in \mathbb{C}$. It is clearly that this is not true for other branch (6.1.1). This also explains why we only consider right inverse in Definition 6.1.1, and we usually consider the principal branch of complex logarithmic (in many literature, we always consider this principal branch unless stated).

PROOF OF THEOREM 6.1.7. We see that

$$\mathrm{Log}\, z = \int_1^z \frac{1}{\zeta}\,\mathrm{d}\zeta = \int_1^{|z|} \frac{1}{\zeta}\,\mathrm{d}\zeta + \int_{|z|}^z \frac{1}{\zeta}\,\mathrm{d}\zeta = \log R + \int_R^{Re^{\mathbf{i}\theta}} \frac{1}{\zeta}\,\mathrm{d}\zeta.$$

We now choose the curve $\mathcal{C} = \left[\, Re^{\mathbf{i}t} \,\middle|\, 0 \leq t \leq \theta \,\right]$, and by the definition of line integral we see that

$$\int_R^{Re^{\mathbf{i}\theta}} \frac{1}{\zeta}\,\mathrm{d}\zeta = \int_{\mathcal{C}} \frac{1}{\zeta}\,\mathrm{d}\zeta = \int_0^\theta \frac{1}{Re^{\mathbf{i}t}} R\mathbf{i}e^{\mathbf{i}t}\,\mathrm{d}t = \mathbf{i}\int_0^\theta 1\,\mathrm{d}t = \mathbf{i}\theta,$$

which conclude the theorem. $\square$

EXERCISE 6.1.9. Show that $\mathrm{Log}\,(z_1 z_2) = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$ for all $z_1, z_2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}$.

EXAMPLE 6.1.10. We now can define the roots of complex number by using $\mathrm{Log}\, z$. For example,

$$\text{the principal branch of } \sqrt{z} := \exp\left(\frac{1}{2}\mathrm{Log}\, z\right) \quad \text{for all } z \in \mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}\, z \leq 0\}.$$

Note that different branches of $\log z$ may yield different branches of $\sqrt{z}$. Unlike $\log z$, there are only two different branches of $\sqrt{z}$. This follows from the fact that the equation $w^2 = z$ has exactly two different solutions for any $z \neq 0$, which is a consequence of fundamental theorem of algebra (Theorem 4.3.6).

EXERCISE 6.1.11. Find all the two branches of $\sqrt{\mathbf{i}}$.

EXAMPLE 6.1.12. The same technique may be used to define arbitrary powers of any nonzero complex number. For example, the principal branch of $\mathbf{i}^{\mathbf{i}}$ is defined by $\exp(\mathbf{i}\mathrm{Log}\,\mathbf{i})$. By using Theorem 6.1.7, one sees that

$$\mathrm{Log}\,\mathbf{i} = \log 1 + \mathbf{i}\frac{\pi}{2} = \frac{\mathbf{i}\pi}{2},$$

then $\mathbf{i}^{\mathbf{i}} = \exp(\mathbf{i}\frac{\mathbf{i}\pi}{2}) = \exp(-\frac{\pi}{2})$. It is interesting to note that $\mathbf{i}^{\mathbf{i}}$ is a real number.

EXERCISE 6.1.13. Determine all the other branches of $\mathbf{i}^{\mathbf{i}}$.

EXERCISE 6.1.14. Compute $\mathrm{Log}\,(1 + \mathbf{i})$.

EXERCISE 6.1.15. Show that

$$\mathrm{Log}\,(1 + z) = -\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} \quad \text{for all } z \in B_1.$$

We end this section by giving an example which has interesting branches which are different to the principal branch.

EXAMPLE (Lambert W-function). The Lambert W-function $W(z)$ is the complex-valued solution of the equation

$$We^W = z.$$

On the z-interval $[0, \infty)$ there is one real solution, and it is nonnegative and increasing. On the z-interval $(-e^{-1}, 0)$, there are two real solutions, one increasing and the other decreasing. We call the increasing solution for which $W(x) \geq W(-e^{-1}) = -1$ the principal branch and denote it by $W_0(x)$, and the decreasing solution can be identified as $W_{\pm 1}(x \mp \mathbf{i}0)$, see Figure 6.1.1. Here $x \mp \mathbf{i}0$ means the (formal) limit $x \mp \mathbf{i}y$ as $y \to 0_+$. Rather than elaborate all details here, we refer to `DLMF:4.13` for more details about this function.

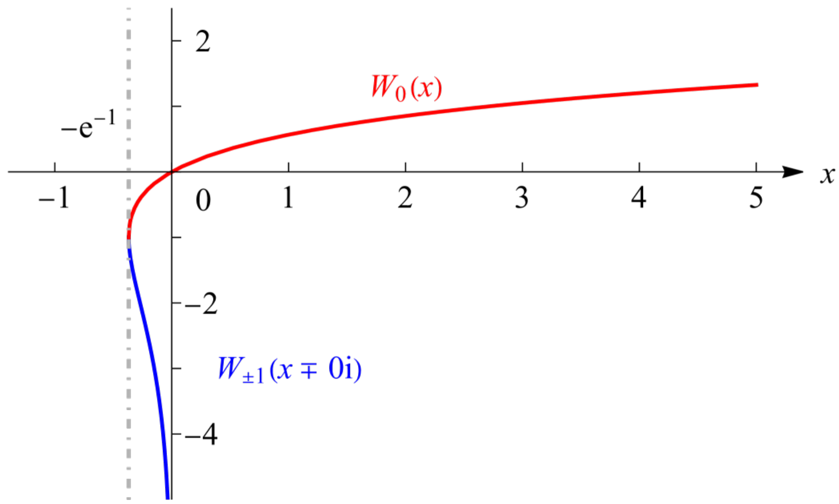


FIGURE 6.1.1. Branches $W_0(x)$, $W_\pm(x \mp \mathbf{i}0)$ of the Lambert W-function (Credit: https://dlmf.nist.gov/4.13.F1.mag)

6.2. Infinite products and Weierstrass product theorem

Similar to the infinite sum (power series), we also can consider the infinite product by using a similar manner:

DEFINITION 6.2.1. Let $\{u_k\}_{k=1}^\infty$ be a sequence of nonzero complex numbers. The infinite product $\Pi_{k\in\mathbb{N}} u_k \equiv \Pi_{k=1}^\infty u_k$ is said to *converge to a nonzero limit* if the sequence of partial products

$$P_N := \Pi_{k=1}^N u_k = u_1 u_2 \cdots u_N$$

converges to a nonzero limit (in $\mathbb{C}$, in the sense of Definition 1.2.4) as $N \to \infty$.

REMARK 6.2.2. In this case, it is easy to see that $P_N = u_N P_{N-1}$. The infinite product converges means $P_N \to P$ for some $P \in \mathbb{C} \setminus \{0\}$, and thus

$$\lim_{N\to\infty} u_N = \lim_{N\to\infty} \frac{P_N}{P_{N-1}} = \frac{\lim_{N\to\infty} P_N}{\lim_{N\to\infty} P_{N-1}} = \frac{P}{P} = 1.$$

Obviously, $\Pi_{k=1}^\infty u_k$ converges to a nonzero limit if and only if $\Pi_{k=N_0}^\infty u_k$ converges to a nonzero limit for any fixed $N_0 \in \mathbb{N}$.

DEFINITION 6.2.3. If $P_N \to 0$, we say the infinite product *diverges to zero*. If there are finitely many terms u_k are equal to zero and $\Pi_{k\in\mathbb{N},u_k\neq 0} u_k$ converges (in $\mathbb{C}$), then se say the product $\Pi_{k\in\mathbb{N}} u_k \equiv \Pi_{k=1}^\infty u_k$ *converges to zero*.

We now give an example to explain why we introduce the term “diverges to zero”.

EXAMPLE 6.2.4. Fix any $N_0 \in \mathbb{N}$, we see that the partial sum of the series $\prod_{k=N_0}^{\infty}(1-1/k)$ is given by

$$\begin{aligned} P_N &:= \prod_{k=N_0}^{N}\left(1-\frac{1}{k}\right) = \prod_{k=N_0}^{N}\frac{k-1}{k} \\ &= \frac{N_0-1}{\cancel{N_0}}\cdot\frac{\cancel{N_0}}{\cancel{N_0+1}}\cdot\frac{\cancel{N_0+1}}{\cancel{N_0+2}}\cdot\cdots\cdot\frac{\cancel{N-1}}{N} \\ &= \frac{N_0-1}{N}. \end{aligned}$$

According to Definition 6.2.1, one has

$$\prod_{k=N_0}^{\infty}\left(1-\frac{1}{k}\right) := \lim_{N\to\infty}\prod_{k=N_0}^{N}\left(1-\frac{1}{k}\right) = 0. \tag{6.2.1}$$

However, we see that $1-\frac{1}{k}\to 1$ as $k\to\infty$. Fix a large N_0, and we formally see that

$$\prod_{k=N_0}^{\infty}\left(1-\frac{1}{k}\right) = \overbrace{\left(1-\frac{1}{N_0}\right)}^{\approx 1}\overbrace{\left(1-\frac{1}{N_0+1}\right)}^{\approx 1}\cdots \overset{(?)}{\approx} 1 \neq 0.$$

Due to this inconsistency, therefore we call (6.2.1) that the series $\prod_{k=N_0}^{\infty}(1-1/k)$ is *diverges* to zero.

EXERCISE 6.2.5. Prove that $\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)$ converges to a nonzero limit.

EXERCISE 6.2.6. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of *positive* real numbers. Show that

$$a_1+a_2+\cdots+a_N \leq \prod_{k=1}^{N}(1+a_k) \leq e^{a_1+a_2+\cdots+a_N} \quad \text{for all } N\in\mathbb{N}.$$

By using this, show that $\prod_{k=1}^{\infty}(1+a_k)$ converges to a nonzero limit if and only if $\sum_{k=1}^{\infty}a_k$ converges.

However, the following exercise demonstrates the necessity of the positivity of such $\{a_k\}_{k=1}^{\infty}$:

EXERCISE 6.2.7. Let $a_k := \frac{(-1)^k}{\sqrt{k}}$ for all $k=2,3,4,\cdots$. Show that $\sum_{k=2}^{\infty}a_k$ converges, but $\prod_{k=2}^{\infty}(1+a_k)$ diverges to zero.

For general (complex) case, we still have the following result.

THEOREM 6.2.8. *Let $1+z_k \in \mathbb{C}\setminus\{z\in\mathbb{C} : \mathfrak{Re}\,z\leq 0\}$ for all $k\in\mathbb{N}$.*

(a) *If $\sum_{k=1}^{\infty}\mathrm{Log}\,(1+z_k)$ converges, then $\prod_{k=1}^{\infty}(1+z_k)$ converges to a nonzero limit $\exp\left(\sum_{k=1}^{\infty}\mathrm{Log}\,(1+z_k)\right)$.*
(b) *If $\prod_{k=1}^{\infty}(1+z_k)$ converges to a nonzero limit, then $\sum_{k=1}^{\infty}\mathrm{Log}\,(1+z_k)$ converges.*

REMARK. The tricky part in (b) is when the limit of $\prod_{k=1}^{\infty}(1+z_k)$ is in $\{z\in\mathbb{C} : \mathfrak{Re}\,z\leq 0\}$, therefore the limit cannot express in terms of the standard logarithmic branches (6.1.1). Therefore the limit of $\sum_{k=1}^{\infty}\mathrm{Log}\,(1+z_k)$ is actually $\log^*\left(\prod_{k=1}^{\infty}(1+z_k)\right)$, where $\log^*$ is some branch of the logarithm given in Theorem 6.1.3 with some suitable domain D, which may differ with the standard choice $\mathbb{C}\setminus\{z\in\mathbb{C} : \mathfrak{Re}\,z\leq 0\}$.

REMARK. Both results (a) and (b) can be extended for all $z_k \neq -1$ with different branches $\log_{(k)}$ (corresponding to different domains $D_{(k)}$) of complex logarithmic for each k.

PROOF OF (a). Let $S_N = \sum_{k=1}^N \mathrm{Log}\,(1+z_k)$ and $P_N = \prod_{k=1}^N (1+z_k) = e^{S_N}$. The condition $\sum_{k=1}^\infty \mathrm{Log}\,(1+z_k)$ converges (to some $S \in \mathbb{C}$) means $S_N \to S$, and hence $P_N \to e^S \neq 0$, which conclude (a). □

PROOF OF (b). The condition $\prod_{k=1}^\infty (1+z_k)$ converges to some nonzero limit $P \in \mathbb{C}$ means $P_N \to P$. As explained in the remark, one can find some branch of the complex logarithm $\log^{**}$ (given in Theorem 6.1.3 with some suitable domain D) such that

$$\log^{**} P_N \to \log^{**} P \text{ in } \mathbb{C} \quad \text{as } N \to \infty.$$

By using Theorem 6.1.3 and (6.1.1), for each $k \in \mathbb{Z}$, one can find $n_k \in \mathbb{Z}$ such that

$$\sum_{k=1}^N (\mathrm{Log}\,(1+z_k) + 2\pi \mathbf{i} n_k) = \log^{**} P_N,$$

and thus

$$\sum_{k=1}^N (\mathrm{Log}\,(1+z_k) + 2\pi \mathbf{i} n_k) \to \log^{**} P \quad \text{as } N \to \infty.$$

It is easy to verify that (this can be showed by, e.g. a contradiction argument)

$$\mathrm{Log}\,(1+z_k) + 2\pi \mathbf{i} n_k \to 0 \quad \text{as } k \to \infty.$$

By using Remark 6.2.2, we have $z_k \to 0$, and thus the above limit implies $n_k \to 0$ as $k \to \infty$. Since $n_k \in \mathbb{Z}$, thus $n_k = 0$ for all $k \geq N_0$ for some $N_0 \in \mathbb{N}$. Then one sees that

$$\sum_{k=1}^\infty \mathrm{Log}\,(1+z_k) = \sum_{k=1}^\infty (\mathrm{Log}\,(1+z_k) + 2\pi \mathbf{i} n_k) \to \log^{**} P + 2\pi \mathbf{i} \sum_{k=1}^{N_0} n_k \quad \text{as } N \to \infty,$$

which proves (b) with the branch $\log^* = \log^{**} + 2\pi \mathbf{i} \sum_{k=1}^{N_0} n_k$. □

COROLLARY 6.2.9. *If $\sum_{k=1}^\infty z_k$ converges absolutely, that is, $\sum_{k=1}^\infty |z_k| < \infty$, then $\prod_{k=1}^\infty (1+z_k)$ converges.*

PROOF. Since $\sum_{k=1}^\infty |z_k| < \infty$, then one can find $N_0 \in \mathbb{N}$ such that $|z_k| < \frac{1}{2}$ for all $k \geq N_0$. Hence by Exercise 6.1.15, one has

$$|\mathrm{Log}\,(1+z_k)| = \left| z_k - \frac{z_k^2}{2} + \frac{z_k^3}{3} - + \cdots \right| \leq |z_k| \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \leq 2|z_k| \quad \text{for all } k \geq N_0.$$

Hence

$$\sum_{k=N_0}^\infty |\mathrm{Log}\,(1+z_k)| \leq 2 \sum_{k=N_0}^\infty |z_k| < \infty$$

and our result follows from Theorem 6.2.8(a). □

DEFINITION 6.2.10. We say that the product $\Pi_{k=1}^\infty (1+z_k)$ is absolutly convergent if $\Pi_{k=1}^\infty (1+|z_k|) < \infty$.

LEMMA 6.2.11. *If $\Pi_{k=1}^\infty (1+z_k)$ is absolutly convergent, then $\prod_{k=1}^\infty (1+z_k)$ converges.*

PROOF. Since $\Pi_{k=1}^\infty (1+|z_k|) < \infty$, by Exercise 6.2.6, we have $\sum_{k=1}^\infty |z_k| < \infty$,. Hence we conclude our lemma by Corollary 6.2.9. □

We wish to consider analytic functions defined by infinite products, i.e. functions of the form

$$f(z) = \prod_{k=1}^{\infty}(1 + u_k(z)). \tag{6.2.2}$$

By using Morera's uniform convergence theorem (Theorem 4.7.5), if each u_k are analytic on an open set D and the partial products converges to their limit function uniformly on each compact set K in D, then one sees that f is analytic on D.

EXERCISE 6.2.12. Let K be a compact set in $\mathbb{C}$, and we consider a continuous function $g : K \to \mathbb{C}$. Show that the set $g(K) := \{g(z) : z \in K\}$ is compact in $\mathbb{C}$.

Based on this observation, one can prove the following theorem.

THEOREM 6.2.13. *Suppose that for each $k = 1, 2, \cdots$ that u_k is analytic in an open set D, and that $\sum_{k=1}^{\infty} |u_k(z)|$ converges uniformly on all compact set in D. Then the product (6.2.2) converges uniformly on on all compact set in D, and it defines an analytic function in D.*

REMARK. The uniform convergence of $\sum_{k=1}^{\infty} u_k(z)$ does not imply the uniform convergence of $\sum_{k=1}^{\infty} |u_k(z)|$.

PROOF. Let K be any compact set in D. Since $\sum_{k=1}^{\infty} |u_k(z)|$ converges uniformly on K, then there exists $N_0 \in \mathbb{N}$ such that $\|u_k\|_{L^\infty(K)} \leq \frac{1}{2}$, hence $1 + u_k \neq 0$ for all $k \geq N_0$. Given any $\epsilon > 0$, one can choose integer $N_1 \geq N_0$ such that

$$\sum_{k=N_1}^{\infty} |u_k(z)| \leq \epsilon.$$

Hence by Exercise 6.1.15, one has

$$\begin{aligned}
|\mathrm{Log}\,(1 + u_k(z))| &= \left| u_k(z) - \frac{(u_k(z))^2}{2} + \frac{(u_k(z))^3}{3} - + \cdots \right| \\
&\leq |u_k(z)| \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \leq 2|u_k(z)| \quad \text{for all } k \geq N_1,
\end{aligned}$$

and thus

$$\sum_{k=N_1}^{\infty} |\mathrm{Log}\,(1 + u_k(z))| \leq 2 \sum_{k=N_1}^{\infty} |u_k(z)| \leq 2\epsilon \quad \text{for all } z \in K.$$

Hence we know that $\sum_{k=1}^{\infty} \mathrm{Log}\,(1+u_k(z))$ converges uniformly on K to a limit function $g(z)$. Sicne g is continuous, by Exercise 6.2.12 it follows that $g(K) := \{g(z) : z \in K\}$ is bounded. Finally, since the exponential function is uniformly continuous in any bounded domain, then

$$P_N(z) := \exp\left(\sum_{k=1}^{N} \log(1 + u_k(z)) \right)$$

converges uniformly to its limit function $f(z) = e^{g(z)}$. Hence we conclude our theorem by the above observation involving Morera's uniform convergence theorem (Theorem 4.7.5). □

EXERCISE 6.2.14. Show that $\prod_{k=1}^{\infty}(1 + z^k)$ converges uniformly on any compact subset of B_1 (therefore it defines an analytic function on B_1).

EXERCISE 6.2.15. Show that $\prod_{k=1}^{\infty}(1+\frac{1}{k^z})$ converges uniformly on any compact subset of the half-space $\{z \in \mathbb{C} : \mathfrak{Re}\,(z) > 1\}$ (therefore it defines an analytic function on $\{z \in \mathbb{C} : \mathfrak{Re}\,(z) > 1\}$).

THEOREM 6.2.16 (Weierstrass product theorem). *Suppse $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ which $|\lambda_k| \to \infty$ as $k \to \infty$. Then there exists an entire function f such that*

$$f(\lambda_k) = 0 \text{ for all } k \in \mathbb{N} \quad f(z) \neq 0 \text{ for all } z \notin \{\lambda_k\}_{k\in\mathbb{N}}.$$

(*see* (6.2.4) *for the precise formula for such* f)

REMARK. According to the uniqueness theorem (Corollary 4.6.7), a nontrivial entire function cannot have an accumulation point of zeros. This means that, if $f(\lambda_k) = 0$ for those $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ converges in $\mathbb{C}$, then $f \equiv 0$ in the whole complex plane $\mathbb{C}$. Therefore the assumption $|\lambda_k| \to \infty$ as $k \to \infty$ seems necessary. It would seem natural to write $f(z) = \prod_{k=1}^{\infty}(z - \lambda_k)$. However, since $|\lambda_k| \to \infty$, the terms of the product would not approach 1, even pointwisely, for each $z \in \mathbb{C}$. The product would diverge.

REMARK. An entire function may be zero at all the points of a sequence which diverges to ∞, see Example 4.6.8 for $\sin z$. Weierstrass product theorem (Theorem 6.2.16) shows that this example is in no way exceptional.

PROOF OF THEOREM 6.2.16. We first consider the case when $\lambda_k \neq 0$ for all $k = 2, 3, \cdots$, and set

$$E_k(z) := \exp\left(\frac{z}{\lambda_k} + \frac{z^2}{2\lambda_k^2} + \cdots + \frac{z^k}{k\lambda_k^k}\right).$$

Given any $M > 0$, and let $|z| < M$. Since $|\lambda_k| \to \infty$, one can find $N_0 \in \mathbb{N}$ such that $|\lambda_k| \geq 2M$ for all $k \geq N_0$. By using Exercise 6.1.15, we see that

$$\begin{aligned}
\mathrm{Log}\left(\left(1 - \frac{z}{\lambda_k}\right)E_k(z)\right) &= \mathrm{Log}\left(1 - \frac{z}{\lambda_k}\right) + \mathrm{Log}\,(E_k(z)) \\
&= \mathrm{Log}\left(1 - \frac{z}{\lambda_k}\right) + \frac{z}{\lambda_k} + \frac{z^2}{2\lambda_k^2} + \cdots + \frac{z^k}{k\lambda_k^k} \\
&= -\sum_{j=k+1}^{\infty}\frac{1}{j}\left(-\frac{z}{\lambda_j}\right)^j
\end{aligned}$$

which is valid since $|\frac{z}{\lambda_k}| \leq \frac{1}{2}$ for all $k \geq N_0$. Hence

$$\begin{aligned}
\left|\mathrm{Log}\left(\left(1 - \frac{z}{\lambda_k}\right)E_k(z)\right)\right| &\leq \sum_{j=k+1}^{\infty}\left|\frac{z^j}{j\lambda_k^j}\right| \\
&= \left|\frac{z}{\lambda_k}\right|^k \sum_{j=k+1}^{\infty}\left|\frac{z^{j-k}}{j\lambda_k^{j-k}}\right| = \left|\frac{z}{\lambda_k}\right|^k \sum_{\ell=1}^{\infty}\frac{1}{\ell+k}\left|\frac{z}{\lambda_k}\right|^{\ell} \\
&\leq \left|\frac{z}{\lambda_k}\right|^k \sum_{\ell=1}^{\infty}\frac{1}{2^{\ell}} = \left|\frac{z}{\lambda_k}\right|^k \leq \frac{1}{2^k}.
\end{aligned}$$

This shows that the sum

$$\sum_{k=N_0}^{\infty}\mathrm{Log}\left(\left(1 - \frac{z}{\lambda_k}\right)E_k(z)\right)$$

is uniform converges in all compact set in B_M. By taking the exponential in each partial sum, one also can verify that the product

$$g(z) := \prod_{k=2}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) E_k(z) \tag{6.2.3}$$

is also uniform converges in all compact set in B_M. By arbitrariness of M, in fact (6.2.3) defines an entire function, satisfying

$$g(\lambda_k) = 0 \text{ for all } k \in \mathbb{N} \quad g(z) \neq 0 \text{ for all } z \notin \{\lambda_k\}_{k=2}^{\infty}.$$

Finally, if we seek an entire function with zeros $\lambda_1 = 0$ at the origin as well, we only need to set

$$f(z) = z^p g(z) = z^p \prod_{k=2}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) E_k(z) \tag{6.2.4}$$

so that $\lambda_1 = 0$ is the zero of f with multiplicity p. □

EXAMPLE 6.2.17. By using (6.2.4), it is easy to see that an entire function with zeros at all the points $\lambda_k = \log k$ for all $k \in \mathbb{N}$ is given by

$$f(z) = z \prod_{k=2}^{\infty} \left(1 - \frac{z}{\log k}\right) \exp\left(\frac{z}{\log k} + \frac{z^2}{2(\log k)^2} + \cdots + \frac{z^k}{k(\log k)^k}\right).$$

EXERCISE 6.2.18. Show that

$$f(z) := \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}$$

is an entire function with a single zero at every negative integer $\lambda_k = -k$. In fact, this function is related to Gamma function (will be introduced later in Section 6.3) by the formula

$$f(z) = \frac{e^{-\gamma z}}{\Gamma(z) z}$$

where γ is the Euler constant, see (6.3.4). [Hint: Modifying the ideas in the proof of Theorem 6.2.16.]

EXAMPLE 6.2.19. By using Exercise 6.2.18, it is easy to see that

$$f(z) = z \left(\prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}\right) \left(\prod_{j=1}^{\infty} \left(1 - \frac{z}{j}\right) e^{\frac{z}{j}}\right) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \tag{6.2.5}$$

is an entire function with a single zero at every integer, since the partial $\prod_{k=1}^{N} \left(1 - \frac{z^2}{k^2}\right)$ is the product of the partial sum

$$\prod_{k=1}^{M_1} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \quad \text{and} \quad \prod_{j=1}^{M_2} \left(1 - \frac{z}{j}\right) e^{\frac{z}{j}}$$

with the special case $N = M_1 = M_2$. As an exercise, here we give a direct justification of $z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ without refering Exercise 6.2.18 but only modifying the proof of Theorem 6.2.16: Given any $M > 1$, and let $|z| < M$. By using Exercise 6.1.15, we see that

$$\mathrm{Log}\left(1-\frac{z^2}{k^2}\right) = -\sum_{j=1}^{\infty}(-1)^j\frac{(-\frac{z^2}{k^2})^j}{j} = -\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{z^2}{k^2}\right)^j \quad \text{for all } k \geq \lceil 2^{\frac{3}{2}}M^3\rceil.$$

Hence we see that

$$\left|\mathrm{Log}\left(1-\frac{z^2}{k^2}\right)\right| \leq \sum_{j=1}^{\infty}\left|\frac{z^2}{k^2}\right|^j \leq \sum_{j=1}^{\infty}\frac{1}{2^j k^{\frac{4}{3}j}} \leq \frac{1}{k^{\frac{4}{3}}}\sum_{j=1}^{\infty}\frac{1}{2^j} = \frac{1}{k^{\frac{4}{3}}} \quad \text{for all } |z| < M.$$

This shows that the sum

$$\sum_{k=\lceil 2^{\frac{3}{2}}M^3\rceil}^{\infty}\mathrm{Log}\left(1-\frac{z^2}{k^2}\right)$$

is uniform converges in all compact set in B_M. By taking the exponential in each partial sum, one also can verify that the product

$$g(z) := \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) \tag{6.2.6}$$

is also uniform converges in all compact set in B_M. By arbitrariness of M, in fact such g defines an entire function, satisfying

$$g(k) = 0 \text{ for all } k \in \mathbb{Z}\setminus\{0\} \quad g(z) \neq 0 \text{ for all } z \notin \mathbb{Z}\setminus\{0\}.$$

Finally, we conclude (6.2.5) is our desired analytic function since $f(z) = zg(z)$.

We have the following fact:

THEOREM 6.2.20. *For each $z \in \mathbb{C}$, we have*

$$\frac{\sin \pi z}{\pi} = f(z),$$

where f is the function given in (6.2.5).

We shall skip the proof of the above theorem, since it is too technical. Here we refer to [**BN10**, Proposition 17.8] for a proof. As an immediate consequence, we have:

COROLLARY 6.2.21. *All zeros of $\sin z$ are real (in other words, there is no zeros other than in Example 4.6.8).*

Moreover, we also have the following representation for complex cosine (here we state without proof, see [**BN10**, Exercise 9 in Chapter 17]).

THEOREM 6.2.22. *For each $z \in \mathbb{C}$, we have*

$$\cos \pi z = \prod_{k=0}^{\infty}\left(1-\frac{4z^2}{(2k+1)^2}\right).$$

6.3. The Gamma function: an extension of factorial function

We begin this section by the following lemma.

LEMMA 6.3.1. *Let D be an open set in $\mathbb{C}$. Suppose $\varphi(z,t)$ is a continuous function of $t \in [a,b]$ for fixed z and an analytic function of $z \in D$ for fixed t. Then*

$$f(z) = \int_a^b \varphi(z,t)\,\mathrm{d}t$$

is analytic in D with complex derivative

$$f'(z) = \int_a^b \partial_z \varphi(z,t)\,\mathrm{d}t. \tag{6.3.1}$$

PROOF. Let Γ the boundary of topological closed rectangle in D, each segment is either horizontal (i.e. parallel to real axis) or vertical (i.e. parallel to imaginary axis). By continuity of φ, one sees that $\varphi \in L^1(\Gamma \times (a,b))$. Therefore by Fubini's theorem (for Lebesgue integral), one sees that

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_\Gamma \int_a^b \varphi(z,t)\,\mathrm{d}t\,\mathrm{d}z = \int_a^b \left(\int_\Gamma \varphi(z,t)\,\mathrm{d}z \right) \mathrm{d}t.$$

Since φ is analytic in z, by Cauchy's residual theorem (Theorem 5.3.6) one sees that $\int_\Gamma \varphi(z,t)\,\mathrm{d}z = 0$, and thus $\int_\Gamma f(z)\,\mathrm{d}z = 0$. By arbitrariness of $\Gamma \subset D$, we conclude f is analytic on D by Morera's theorem (Theorem 4.7.1). Since f is analytic, then $f'(z) = \partial_x f$, whenever $z = x + \mathbf{i}y$. Therefore (6.3.1) immediately follows from the Leibniz integral rule (this step only requires the continuity of $\partial_x \varphi$), here we omit the details. $\square$

We consider the integral

$$I_n = \int_0^\infty e^{-t} t^n \,\mathrm{d}t \quad \text{for } n = 0,1,2,\cdots,$$

which can be interpret as improper Riemann integral. In this case, this is same as the Lebesgue integral.

EXERCISE 6.3.2. By interpreting the above as improper Riemann integral, show that $I_0 = 1$ and $I_n = nI_{n-1}$ for all $n \in \mathbb{N}$. From this, one sees that $I_n = n! = n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1$.

For any $z \in \mathbb{C}$ and $t > 0$, we define $t^{z-1} := e^{(z-1)\log t}$. One sees that $|t^{z-1}| = |e^{(z-1)\log t}| = e^{(\mathfrak{Re}\,(z-1))\log t} = t^{\mathfrak{Re}\,(z-1)}$ for all $t > 0$. Hence one sees that the *gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t$$

is uniformly convergent in the right half-plane $\{z \in \mathbb{C} : \mathfrak{Re}\, z > 0\}$. Hence by Lemma 6.3.1, one sees that Γ is analytic in the right half-plane $\{z \in \mathbb{C} : \mathfrak{Re}\, z > 0\}$ with complex derivatives or order k:

$$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1} (\log t)^k e^{-t}\,\mathrm{d}t \quad \text{for all } z \in \mathbb{C} \text{ with } \mathfrak{Re}\, z > 0.$$

Using the same arguments in Exercise 6.3.2, it is easy to show that

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for all } z \in \mathbb{C} \text{ with } \mathfrak{Re}\, z > 0.$$

We can extend Γ for $-1 < \mathfrak{Re}\, z < 0$ by the formula

$$\Gamma(z) := \frac{\Gamma(z+1)}{z} \quad \text{for all } z \in \mathbb{C} \text{ with } -1 < \mathfrak{Re}\, z < 0.$$

It is easy to see that Γ is continuous at each $z \in \mathbb{C} \setminus \{0\}$ with $\mathfrak{Re}\, z > -1$, and hence by using Morera's continuity theorem (Theorem 4.7.7), $\Gamma$is also analytic there. Continuing in the same manner, we can define

$$\Gamma(z) := \frac{\Gamma(z+2)}{z(z+1)} \quad \text{for all } z \in \mathbb{C} \text{ with } -2 < \mathfrak{Re}\, z < -1,$$
$$\Gamma(z) := \frac{\Gamma(z+3)}{z(z+1)(z+2)} \quad \text{for all } z \in \mathbb{C} \text{ with } -3 < \mathfrak{Re}\, z < -2,$$
$$\Gamma(z) := \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k)} \quad \text{for all } z \in \mathbb{C} \text{ with } -k-1 < \mathfrak{Re}\, z < -k, \tag{6.3.2}$$

and applying Morera's continuity theorem (Theorem 4.7.7), we see that:

THEOREM 6.3.3. Γ *defines an analytic function on* $\mathbb{C}\setminus\{0,-1,-2,\cdots\}$, *with* $\mathrm{Res}\,(\Gamma;-k) = \lim_{z\to -k}(z+k)\Gamma(z) = \frac{(-1)^k}{k!}$.

PROOF. By using Proposition 5.3.2, one can easily compute

$$\mathrm{Res}\,(\Gamma;-k) = \lim_{z\to -k}(z+k)\Gamma(z) = \frac{\Gamma(1)}{(-k)(-k+1)\cdots(-1)} = \frac{(-1)^k}{k!},$$

which concludes our result. This also means that $\{0,-1,-2,\cdots\}$ are all poles or order 1. □

From now on, we will only sketch the ideas (since this part is quite technical), see [**BN10**, Chapter 18] for more details. By using the fact that $\lim_{n\to\infty}(1-\frac{t}{n})^n = e^{-t}$, one can show

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n \mathrm{d}t \\ &= \lim_{n\to\infty}\frac{1}{n^n}\int_0^n t^{z-1}(n-t)^n\,\mathrm{d}t \quad \text{whenever } \mathfrak{Re}\, z > 0,\end{aligned}$$

see [**BN10**, Exercise 7 in Chapter 18]. By using integration by parts, we have

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty}\frac{1}{n^n}\cdot\frac{n}{z}\int_0^n t^z(n-t)^{n-1}\,\mathrm{d}t \\ &= \lim_{n\to\infty}\frac{1}{n^n}\frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)}\int_0^n t^{z+n-1}\,\mathrm{d}t \\ &= \lim_{n\to\infty}\frac{n^z}{z}\cdot\frac{1}{z+1}\cdot\frac{2}{z+2}\cdot\cdots\cdot\frac{n}{z+n}.\end{aligned}$$

Thus we reach the *Gauss' product representation* for Gamma function:

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \lim_{n\to\infty} zn^{-z}(1+z)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right) \\ &= \lim_{n\to\infty} zn^{-z}\prod_{k=1}^n\left(1+\frac{z}{k}\right),\end{aligned} \tag{6.3.3}$$

see also [**FB09**, Proposition IV.1.10].

REMARK. This immediately shows that Γ has no zeros.

Here we exhibit a real analysis result in [**BN10**, Lemma 18.8]:

LEMMA 6.3.4. *If $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$, then $\lim_{n\to\infty} s_n$ exists. This limit is called the Euler constant, usually denoted as γ.*

We write (6.3.3) as

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} e^{z(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)} \left(z \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right).$$

By using Theorem 6.2.20 and Lemma 6.3.4, we have

$$\frac{1}{\Gamma(z)} = e^{\gamma z} \left(z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right) \quad \text{whenever } \mathfrak{Re}\, z > 0, \tag{6.3.4}$$

see Exercise (6.2.18). By using the extension formula (6.3.2), in fact

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right),$$

this somehow formaly replace z by $-z$ (but in fact not so obvious). Therefore, we reach

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z \sin \pi z} \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{Z},$$

that is (see also [**FB09**, Proposition IV.1.11]):

THEOREM 6.3.5 (Completion Formula). $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ *for all* $z \in \mathbb{C} \setminus \mathbb{Z}$.

As an immediate consequence, we have

$$\Gamma(1/2) = \sqrt{\pi}.$$

Applying the identity $\Gamma(z+1) = z\Gamma(z)$, we also have $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(5/2) = 3\sqrt{\pi}/4$, and so on.

EXERCISE 6.3.6. Show that

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \prod_{k=0}^{n-1} \left(k + \frac{1}{2}\right) \quad \text{for all } n = 0, 1, 2, \cdots.$$

We now restrict $\Gamma(z)$ for $z > 0$. In fact, $\log \circ \Gamma$ is convex on $(0, \infty)$, see [**Rud76**, Theorem 8.18]. It is a rather surprising fact (discovered by Borh and Mollerup) that: If f is a positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$, $f(1) = 1$ and $\log \circ f$ is convex, then $f = \Gamma$ on $(0, \infty)$, see [**Rud76**, Theorem 8.19]. See also [**FB09**, Proposition IV.1.3] for a characterization of the complex Γ-function. We finally end this section by exhibit a version of Stirling's formula, which can be found in [**FB09**, Proposition IV.1.14].

THEOREM 6.3.7 (General Stirling's formula). *Let H be the function*

$$H(z) = \sum_{n=0}^{\infty} \left(\left(z + n + \frac{1}{2}\right) \operatorname{Log}\left(1 + \frac{1}{z+n}\right) - 1 \right).$$

Then for all $z \in \mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}\, z \le 0\}$ one has

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} e^{H(z)}.$$

In any angular domain $W_\delta = \left\{z = |z|e^{\mathbf{i}\theta} : -\pi + \delta \leq \theta \leq \pi - \delta\right\}$ *with* $0 < \delta \leq \pi$*, we have* $H(z) \to 0$ *as* $z \to \infty$*. In addition, we have*

$$0 < H(x) < \frac{1}{12x} \quad \text{for all } x > 0.$$

Therefore we have the ordinary Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\varphi(n)}{12n}} \quad \text{with} \quad 0 < \varphi(n) < 1.$$

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