An introduction to partial differential equations and functional analysis
Lecture notes, Spring 2024
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Preface

This lecture note is prepared for the course *Partial Differential Equations*, for undergraduate (701925001) and postgraduate (751944001) levels, during Spring 2024 (112-2). My dissertation [**Kow21**] itself also a lecture note for some advance topics in PDE. The lecture note may updated during the course.

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Acknowledgments. I would like to give special thanks to students who pointed out my mistakes in this note.

Exam rules.

(1) Textbook or other materials are not allowed to be used during exams.
(2) All electronic devices (including calculator, smartphone, pad, computer, ...) are prohibited during exams.
(3) One also not allowed to bring your own extra paper. TA will provide answer sheets.
(4) Before go to washroom, one must inform us before do so.
(5) If you violate one of the above rule, we will immediate terminate your writing and the marks of the exam/quiz will be 0.
(6) One must show student card or national identity card or national health insurance card or passport or resident certificate (driving license not accepted) for verification. Before the exam begins, TA should reminds all of you to bring it. If one fails to show it during exam, we consider this as a cheating and the marks of the quiz will be 0.

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CHAPTER 1

Preliminaries

Throughout this course, we will use Lebesgue integral rather than Riemannian integral. Here we only exhibit some important properties about differentiation and integration that everyone must know, following the presentation in [**Bre11**, Chapter 4], one can refer e.g. the monograph [**WZ15**] for a nice introduction about Lebesgue integral. In order to avoid too much terminology, here we only exhibit the results for open sets Ω, but however they also valid for more general domains. Throughout this lecture note, the abbreviation "a.e." means "almost everywhere", and we usually omit this if there is no ambiguity.

In this lecture note, we will denote the vector[1] by $\boldsymbol{x} := (x_1, x_2, \cdots, x_n)$, despite it means the column vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Let Ω be any open set in $\mathbb{R}^n$, and for each $1 \leq p \leq \infty$ we define

$$L^p(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ with } \|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p} < \infty \right\} \quad \text{when } 1 \leq p < \infty,$$

$$L^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ with } \|f\|_{L^\infty(\Omega)} := \sup_{\boldsymbol{x} \in \Omega} |f(\boldsymbol{x})| < \infty \right\}.$$

For each $1 \leq p < \infty$, one sees that $\| |f|^p \|_{L^1(\Omega)} = \|f\|_{L^p(\Omega)}^p$, therefore in many cases it is suffice to consider L^1-functions. We first collect some properties of L^1 spaces in the following theorem.

LEMMA 1.0.1 (Monotone convergence theorem, Beppo Levi). *Let Ω be an open set in $\mathbb{R}^n$, and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1(\Omega)$ satisfying*

$$f_1(\boldsymbol{x}) \leq f_2(\boldsymbol{x}) \leq \cdots \leq f_k(\boldsymbol{x}) \leq f_{k+1}(\boldsymbol{x}) \leq \cdots \quad \text{for a.e. } \boldsymbol{x} \in \Omega, \quad \sup_{k\in\mathbb{N}} \int_\Omega f_k < \infty,$$

then $f_k(\boldsymbol{x})$ converges a.e. on Ω to a finite limit, which we denote by $f(\boldsymbol{x})$. In addition, such limit function $f(\boldsymbol{x})$ belongs to $L^1(\Omega)$ and satisfies

$$f_k \to f \text{ in } L^1(\Omega), \quad \text{that is,} \quad \lim_{k\to\infty} \|f_k - f\|_{L^1(\Omega)} = 0.$$

LEMMA 1.0.2 (Lebesgue dominated convergence theorem [**Bre11**, Theorem 4.2]). *Let Ω be an open set in $\mathbb{R}^n$ and let $\{f_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ be a sequence of functions satisfying*

(1) *$f_k(\boldsymbol{x}) \to f(\boldsymbol{x})$ a.e. in Ω;*

(2) *there is a function $g \in L^1(\Omega)$ such that $|f_k(\boldsymbol{x})| \leq g(\boldsymbol{x})$ a.e. in Ω for all $k \in \mathbb{N}$.*

Then $f \in L^1(\Omega)$ and $f_k \to f$ in $L^1(\Omega)$.

[1] Some author also use the notation $\vec{x}$, or simply x.

LEMMA 1.0.3 (Fatou's lemma). *Let Ω be an open set in $\mathbb{R}^n$ and let $\{f_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ be a sequence of functions satisfying*

$$f_k(\boldsymbol{x}) \geq 0 \text{ for a.e. } \boldsymbol{x} \in \Omega \text{ and for all } k \in \mathbb{N}, \quad \sup_{k\in\mathbb{N}} \int_\Omega f_k < \infty.$$

We set $f(\boldsymbol{x}) := \liminf_{k\to\infty} f_k(\boldsymbol{x}) \leq +\infty$. Then $f \in L^1(\Omega)$ and

$$\int_\Omega f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \leq \liminf_{k\to\infty} \int_\Omega f_k(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

It is also important to mention the following fact:

THEOREM 1.0.4 (Fubini's theorem). *Let Ω_1 be an open set in $\mathbb{R}^n$ and let Ω_2 be an open set in $\mathbb{R}^m$, and let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a (measurable) function.*

(1) *If $F \geq 0$ a.e. in $\Omega_1 \times \Omega_2$, then*

$$\int_{\Omega_2} \int_{\Omega_1} F(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y} = \int_{\Omega_1} \int_{\Omega_2} F(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{x}. \tag{1.0.1}$$

(2) *If $F \in L^1(\Omega_1 \times \Omega_2)$, i.e. $\int_{\Omega_2} \int_{\Omega_1} |F(\boldsymbol{x}, \boldsymbol{y})|\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y} < \infty$, then (1.0.1) also holds.*

We now collect some elementary properties of L^p spaces in the following theorem.

THEOREM 1.0.5 ([**Bre11**, Theorems 4.6–4.8]). *Let Ω be an open set in $\mathbb{R}^n$. Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1$. Then $fg \in L^1(\Omega)$ and the following* Hölder's inequality *holds:*

$$\int_\Omega |f(\boldsymbol{x}) g(\boldsymbol{x})|\, \mathrm{d}\boldsymbol{x} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)} \tag{1.0.2}$$

and the equality in (1.0.2) holds when there exists $c \in \mathbb{R}$ such that $|g(\boldsymbol{x})| = c|f(\boldsymbol{x})|^{p-1}$ for a.e. $\boldsymbol{x} \in \Omega$. In addition, the function $\|\cdot\|_{L^p(\Omega)} : L^p(\Omega) \to \mathbb{R}$ defines a norm, and $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space for all $1 \leq p \leq \infty$ (this is well-known as Fischer-Riesz Theorem*).*

EXERCISE 1.0.6. For each $1 < p < \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1$, show the following inequality:

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'} \quad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

Use this to conclude the Hölder's inequality (1.0.2). [Hint: One way to show this is using the concavity of the logarithmic function on $(0, \infty)$.]

EXERCISE 1.0.7. Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $h \in L^r(\Omega)$ for some $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Then $fgh \in L^1(\Omega)$ and

$$\int_\Omega |f(\boldsymbol{x}) g(\boldsymbol{x}) h(\boldsymbol{x})|\, \mathrm{d}\boldsymbol{x} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \|h\|_{L^r(\Omega)}.$$

EXERCISE 1.0.8. Show that $\|\cdot\|_{L^p(\Omega)} : L^p(\Omega) \to \mathbb{R}$ defines a norm. [Hint: Use the convexity of the mapping $t \mapsto t^p$ for each $1 \leq p < \infty$]

EXERCISE 1.0.9. Given any $f \in L^p(\Omega)$, show that

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega f(\boldsymbol{x}) g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{1.0.3}$$

as well as

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \tag{1.0.4}$$

EXERCISE 1.0.10 (Minkowski's integral inequality). In fact, (1.0.3) holds true for any (measurable) f (not necessary in $L^p(\Omega)$). Using this fact to show

$$\left(\int_{\Omega_2} \left| \int_{\Omega_1} F(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{x} \right|^p \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}} \leq \int_{\Omega_1} \left(\int_{\Omega_2} |F(\boldsymbol{x}, \boldsymbol{y})|^p \, \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}} \mathrm{d}\boldsymbol{x}.$$

EXERCISE 1.0.11. Deduce that if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, then $f \in L^r(\Omega)$ for every r between p and q. More precisely, write

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad \text{with } 0 \leq \alpha \leq 1$$

and prove that

$$\|f\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}^\alpha \|f\|_{L^q(\Omega)}^{1-\alpha}.$$

In the context of PDE, the following notion plays a central role:

DEFINITION 1.0.12. The *convolution* of two measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is the function $f * g : \mathbb{R}^n \to \mathbb{R}$ given by

$$(f * g)(\boldsymbol{x}) := \int_{\mathbb{R}^n} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

provided that the integral exists a.e. A change of variable gives that $f * g = g * f$.

The convolution is well-defined in the following sense:

LEMMA 1.0.13 (Young's inequality [**Bre11**, Exercise 4.30]). *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Set $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ so that $1 \leq r \leq \infty$. Let $v \in L^p(\mathbb{R}^n)$ and $\rho \in L^q(\mathbb{R}^n)$. Then $\rho * v \in L^r(\mathbb{R}^n)$ and*

$$\|\rho * v\|_{L^r(\mathbb{R}^n)} \leq \|v\|_{L^p(\mathbb{R}^n)} \|\rho\|_{L^q(\mathbb{R}^n)}.$$

We now collect some definitions and facts about differentiation. For a one variable function u, its derivative at $t \in \mathbb{R}$ is (at least formally) defined by

$$u'(t) := \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}.$$

Let $\boldsymbol{e}_j$ be the j^{th}-column of the $n \times n$ identity matrix, and the partial derivatives are (formally) defined by

$$\partial_j u(\boldsymbol{x}) := \lim_{h \to 0} \frac{u(\boldsymbol{x} + h\boldsymbol{e}_j) - u(\boldsymbol{x})}{h}.$$

We recall the following fundamental theorem:

LEMMA 1.0.14 (see e.g. [**Apo74**, Theorem 12.13]). *If both partial derivatives $\partial_{i_1} u$ and $\partial_{i_2} u$ exist* near *a point $\boldsymbol{x}_0$ and if both $\partial_{i_1}\partial_{i_2} u$ and $\partial_{i_2}\partial_{i_1} u$ are continuous* at *$\boldsymbol{x}_0$, then*

$$\partial_{i_1}\partial_{i_2} u(\boldsymbol{x}_0) = \partial_{i_2}\partial_{i_1} u(\boldsymbol{x}_0).$$

Therefore, we usually denote $\partial_{i_1 i_2} = \partial_{i_1}\partial_{i_2} = \partial_{i_2}\partial_{i_1}$, regardless the order of partial derivatives. We also exhibit some notations which are helpful to express higher order derivatives. For each multi-index $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ with non-negative integers α_j, we define

$$\mathrm{supp}\,(\boldsymbol{\alpha}) := \{j \in \{1, \cdots, n\} : \alpha_j \neq 0\}, \quad |\boldsymbol{\alpha}| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \equiv \sum_{j=1}^{n} \alpha_j,$$

as well as

$$\partial^{\boldsymbol{\alpha}} := \prod_{j \in \mathrm{supp}\,(\boldsymbol{\alpha})} \partial_j^{\alpha_j} \quad \text{with the convention} \quad \partial^{(0,\cdots,0)} := \mathrm{Id},$$

and *the partial derivatives in $\partial^{\boldsymbol{\alpha}}$ are pairwise commute.* In view of Lemma 1.0.14, for each open set Ω and a non-negative integer k, we define the spaces

$$\begin{aligned} C^k(\Omega) &:= \{u : \Omega \to \mathbb{C} : \partial^{\boldsymbol{\alpha}} u \text{ is continuous for all } \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| \leq k\}, \\ C^k(\overline{\Omega}) &:= \left\{u|_{\overline{\Omega}} : u \in C^k(U) \text{ for some open set } U \supset \overline{\Omega}\right\}, \\ C_c^k(\Omega) &:= \left\{u \in C^k(\Omega) : \mathrm{supp}\,(u) \subset \Omega \text{ is compact}\right\}. \end{aligned}$$

By considering the zero extension, one also sees that

$$C_c^k(\Omega) = \left\{u \in C^k(\mathbb{R}^n) : \mathrm{supp}\,(u) \subset \Omega \text{ is compact}\right\}. \tag{1.0.5}$$

Similarly, we also write

$$\begin{aligned} C_c^\infty(\Omega) &:= \{u \in C^\infty(\Omega) : \mathrm{supp}\,(u) \subset \Omega \text{ is compact}\} \\ &= \{u \in C^\infty(\mathbb{R}^n) : \mathrm{supp}\,(u) \subset \Omega \text{ is compact}\}. \end{aligned}$$

The following density result is fundamental:

Lemma 1.0.15. *Let Ω be an open set in $\mathbb{R}^n$. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$, that is, given any $f \in L^p(\Omega)$, there exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that $f_k \to f$ in $L^p(\Omega)$.*

Exercise 1.0.16. Show that Lemma 1.0.15 does not hold true when $p = \infty$.

We now state the following proposition, which serves as the most important ingredient in this course:

Proposition 1.0.17 (Divergence theorem, see e.g. [**Str08**, Appendix A.3]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with a piecewise-C^1 boundary $\partial\Omega$. Let $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n)$ be the unit outward normal vector on $\partial\Omega$, then*

$$\int_\Omega \partial_i f \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \nu_i f \,\mathrm{d}S_{\boldsymbol{x}} \tag{1.0.6}$$

for all $f \in C^1(\overline{\Omega})$, where $\mathrm{d}S_{\boldsymbol{x}}$ is the surface element (can be characterized in terms of Hausdorff measure) on $\partial\Omega$.

Exercise 1.0.18. Suppose that all assumptions in Proposition 1.0.17 holds. Show that

$$\int_\Omega \nabla \cdot \boldsymbol{f} \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{\nu} \cdot \boldsymbol{f} \,\mathrm{d}S_{\boldsymbol{x}} \tag{1.0.7}$$

for all $\boldsymbol{f} = (f_1, \cdots, f_n) \in (C^1(\overline{\Omega}))^n$, where $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) := \partial_1 f_1(\boldsymbol{x}) + \cdots + \partial_n f_n(\boldsymbol{x})$ is called the *divergence* of f.

Using product rule, it is easy to see that

$$\int_{\partial\Omega} \nu_i f g \,\mathrm{d}S_{\boldsymbol{x}} = \int_{\Omega} \partial_i (fg) \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \partial_i f(\boldsymbol{x}) g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} + \int_{\Omega} f(\boldsymbol{x}) \partial_i g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

Hence we immediately reach the following useful corollary:

COROLLARY 1.0.19 (Integration by parts). *Suppose that all assumptions in Proposition 1.0.17 holds, then*

$$\int_{\Omega} \partial_i f(\boldsymbol{x}) g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \nu_i(\boldsymbol{x}) f(\boldsymbol{x}) g(\boldsymbol{x}) \,\mathrm{d}S_{\boldsymbol{x}} - \int_{\Omega} f(\boldsymbol{x}) \partial_i g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$$

for all $f, g \in C^1(\overline{\Omega})$.

REMARK. We now restrict ourselves when $n = 1$. When $\Omega = (a, b)$, the above identity simply reads

$$\int_a^b f'(x) g(x) \,\mathrm{d}x = f(x) g(x)|_{x=a}^{x=b} - \int_a^b f(x) g'(x) \,\mathrm{d}x$$

which is just the usual integration by parts. In fact, each open set Ω can be written as union of countably many disjoint open intervals [**WZ15**], and therefore the above formula can be extended for arbitrary open sets in $\mathbb{R}^1$. The fundamental theorem of calculus is simply a special case of divergence theorem. Indeed Corollary 1.0.19 can be extended for general bounded Lipschitz domains and in weak sense (see Theorem 3.2.8 below).

EXERCISE 1.0.20 (Green's theorem as a special case of divergence theorem). Suppose that all assumptions in Proposition 1.0.17 holds with $n = 2$. The Green's theorem stated that

$$\int_{\Omega} (\partial_x q(x, y) - \partial_y p(x, y)) \,\mathrm{d}x \,\mathrm{d}y = \int_{\partial D} (p \,\mathrm{d}x + q \,\mathrm{d}y),$$

for all $p, q \in C^1(\overline{\Omega})$, where the right-hand-side is the line integral defined by

$$\int_{\partial D} (p \,\mathrm{d}x + q \,\mathrm{d}y) := \int_{\partial D} (p(\boldsymbol{\gamma}(s)), q(\boldsymbol{\gamma}(s))) \cdot \boldsymbol{t}(s) \,\mathrm{d}s$$

where $\boldsymbol{\gamma}(s)$ is the arc-lengh parametrization of ∂D and $\boldsymbol{t}(s)$ is the unit tangent vector field (usually chosen to be counterclockwise oriented). Show that Green's theorem is a special case of the divergence theorem.

CHAPTER 2

Partial differential equation in classical sense

2.1. What is partial differential equations

In many cases, it is not convenient to write down the function explicitly. For example, for the function $u(t) := \sin^{-1} t$ for $-1 < t < -1$, it is more convenient to write it as

$$\sin(u(t)) = t.$$

Taking derivative on both sides of the above equation, or in some fancy words "performing implicit differentiation", we reach

$$\cos(u(t))u'(t) = 1.$$

If we write $F(t, u, u') := \cos(u(t))u'(t) - 1$, then we see that the above equation is simply a special case of the following *first-order ordinary differential equation*:

$$F(t, u, u') = 0.$$

In general, for any $k \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \cdots\}$, the most general $k^{\rm th}$*-order ordinary differential equation* (*ODE*) takes the form

$$F(t, u, u', \cdots, .u^{(k)}) = 0,$$

where $u^{(k)}$ is the $k^{\rm th}$-derivative of u. The key defining property of a partial differential equation is that there is more than one independent variable $x_1, x_2, \cdots, x_n$ ($n \in \mathbb{N}$). Similar as above, we now introduce the following definition.

Definition 2.1.1. The general $k^{\rm th}$*-order partial differential equation* (*PDE*) takes the form

$$F\left(\boldsymbol{x}, \{\partial^{\boldsymbol{\alpha}} u\}_{|\boldsymbol{\alpha}|\leq k}\right) \equiv F\left(\boldsymbol{x}, u, \{\partial^{\boldsymbol{\alpha}} u\}_{|\boldsymbol{\alpha}|=1}, \cdots, \{\partial^{\boldsymbol{\alpha}} u\}_{|\boldsymbol{\alpha}|=k}\right) = 0. \tag{2.1.1}$$

A solution of (2.1.1) is a function u that satisfies the equation identically in some region (open sets) in $\mathbb{R}^n$.

In view of Theorem 1.0.14, the above definition is at least well-defined for C^k-solutions u. It is convenient to write the PDE in operator form: We write

$$\mathcal{L}u := F\left(\boldsymbol{x}, \{\partial^{\boldsymbol{\alpha}} u\}_{|\boldsymbol{\alpha}|\leq k}\right),$$

where F is the function given in (2.1.1).

Definition 2.1.2. Given any function g, and we consider a PDE $\mathcal{L}u = g$. If $g \equiv 0$, then we say that the PDE is *homogeneous*. We say that the PDE is:

(1) *linear* when $(\mathcal{L}u)(\boldsymbol{x}) = \displaystyle\sum_{|\boldsymbol{\alpha}|\leq k} c_{\boldsymbol{\alpha}}(\boldsymbol{x})\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x})$ for some functions $c_{\boldsymbol{\alpha}}$;

(2) *semilinear* when $(\mathcal{L}u)(\boldsymbol{x}) = \displaystyle\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}}(\boldsymbol{x})\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x}) + G\left(\boldsymbol{x}, \{\partial^{\boldsymbol{\alpha}} u\}_{|\boldsymbol{\alpha}|<k}\right)$;

(3) *quasilinear* when $(\mathcal{L}u)(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}}\left(\boldsymbol{x}, \{\partial^{\boldsymbol{\alpha}}u\}_{|\boldsymbol{\alpha}|<k}\right)\partial^{\boldsymbol{\alpha}}u(\boldsymbol{x})$.

If $\mathcal{L}$ is semilinear, sometimes we refer

$$(\mathcal{L}_{\mathrm{prin}}u)(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}}(\boldsymbol{x})\partial^{\boldsymbol{\alpha}}u(\boldsymbol{x})$$

the *principal part* of $\mathcal{L}$. Here are some examples:

(1) The transport equation $\sum_{|\boldsymbol{\alpha}|=1}\partial^{\boldsymbol{\alpha}}u = 1$ is a first order linear PDE.
(2) The shock wave equation $\partial_1 u + u\partial_2 u = 0$ is a first order quasilinear PDE.
(3) The Laplace equation $\partial_1^2 u + \cdots \partial_n^2 u = 0$ is a second order linear PDE. We often denote the *Laplace operator* (or *Laplacian*) by $\Delta u = 0$. Here "Δ" is the capital-delta in Greek.
(4) The diffusion/heat/caloric equation $\partial_t u - \Delta u = 0$ is a second order linear PDE.
(5) The wave equation $\partial_t^2 u - \Delta u = 0$ is a second order linear PDE.
(6) The Schrödinger equation $\partial_t u - \mathbf{i}\Delta u = 0$, with the imaginary number $\mathbf{i} = \sqrt{-1}$ [**BN10**, **FB09**, **Kow23**], is a second order linear PDE.
(7) The dispersive wave equation $\partial_t u + u\partial_x u + \partial_x^3 u = 0$ is a third order semilinear PDE.
(8) The Korteweg-deVries (KdV) equation $\partial_t u + \partial_x^3 u + 6u\partial_x u = 0$ is a third order semilinear PDE.

REMARK. It is still possible to generalize the PDE in Definition 2.1.1, for example, from the *pseudodifferential* operator point of view. For example, the fractional Laplacian, see e.g. [**Kwa17**] for (at least) ten equivalent definition for fractional Laplacian, or my PhD dissertation [**Kow21**] for fractional order elliptic equations. Here we also refer to a monograph [**KRY20**] for a nice introduction on Riemann-Liouville/Caputo derivatives in weak sense. We will not cover these advance topics in this lecture note.

2.2. First order PDE

2.2.1. Transport equation. We begin our discussion of PDEs by solving some simple ones. Given a horizontal pipe of fixed cross section in the (positive) x-direction. Suppose that there is a fluid flowing at a constant rate c ($c = 0$ means the fluid is stationary; $c > 0$ means flowing toward right, otherwise towards left). We now assume that there is a substance is suspended in the water.

Fix a point at the pipe, and we set the point as the origin 0, and let $u(t,x)$ be the concentration of such substance. The amount of pollutant in the interval $[0,y]$ at time t is given by

$$\int_0^y u(t,x)\,\mathrm{d}x.$$

At the later time $t+\tau$, the same molecules of pollutant moved by the displacement $c\tau$, and this means

$$\int_0^y u(t,x)\,\mathrm{d}x = \int_{c\tau}^{y+c\tau} u(t+\tau,x)\,\mathrm{d}x.$$

If u is continuous, by using the fundamental theorem of calculus, by differentiating the above equation with respect to y, one sees that

$$u(t,y) = u(t+\tau, y+c\tau) \quad \text{for all } y \in \mathbb{R}. \tag{2.2.1}$$

If we further assume $u \in C^1$, then differentiating (2.2.1) with respect to τ, we reach the following *transport equation*:

$$0 = u(t+\tau, y+c\tau)|_{\tau=0} = \partial_t u(t,x) + c\partial_x u(t,x) \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}.$$

2.2.2. The constant coefficient equation. The simplest possible PDE is $\partial_t u(t,x) = 0$. Its general solution is $u(t,x) = f(x)$, where f is any function of one variables. Because the solutions are independent of t, they are constant on the lines $x =$ constant in the (t,x)-plane. Let $c =$ constant $\neq 0$ and let us solve the transport equation

$$\partial_t u + c\partial_x u = 0 \tag{2.2.2}$$

for a function $u = u(t,x) \in C^1(\mathbb{R} \times \mathbb{R})$. Given any $\xi \in \mathbb{R}$, we see that the set L_ξ of points (t,x) solving $x - ct = \xi$ is a straight line in (t,x)-plane. Since we see that

$$\partial_t \left(u|_{L_\xi}(t) \right) = \partial_t \left(u(t, ct+\xi) \right) = (\partial_t u + c\partial_x u)|_{L_\xi}(t) = 0 \quad \text{for all } t \in \mathbb{R},$$

then by using chain rule we see that

$$u|_{L_\xi}(t) = u(t, ct+\xi) = u(0,\xi) = u(0, x-ct) \quad \text{for all } t \in \mathbb{R}.$$

Hence it is make sense to refer L_ξ the *characteristic line* of (2.2.2). This means that the general solution of (2.2.2) must takes the form

$$u(t,x) = f(x-ct) \tag{2.2.3}$$

for some function $f \in C^1(\mathbb{R})$. Formula (2.2.3) represents the general solution u *uniquely* in terms of its *initial values*

$$u(0,x) = f(x).$$

In other words, if u_1 and u_2 are $C^1(\mathbb{R})$ solutions of (2.2.2) with $u_1(0,x) = u_2(0,x) = f(x)$, then $u_1 \equiv u_2$ throughout the whole (t,x)-plane. Conversely, every u of the form (2.2.3) is a solution of (2.2.2) with initial values f provided f is of class $C^1(\mathbb{R})$. We conclude the following in the following theorem:

THEOREM 2.2.1. *Given any $f \in C^1(\mathbb{R})$, there exists a unique solution $u \in C^1(\mathbb{R}^2)$ of (2.2.2). In addition, the solution is of the form (2.2.3).*

REMARK. Here we also exhibit another way to compute the solutions. By writing $t' = t + cx$ and $x' = ct - x$, and abusing the notation $u(t,x)$ and $u(t',x')$, by using chain rule one sees that

$$\begin{aligned} \partial_t u &= \partial_{t'} u \partial_t t' + \partial_{x'} u \partial_t x' = \partial_{t'} u + c\partial_{x'} u, \\ \partial_x u &= \partial_{t'} u \partial_x t' + \partial_{x'} u \partial_x x' = c\partial_{t'} u - \partial_{x'} u. \end{aligned}$$

Together with (2.2.2), one has

$$(1+c^2)\partial_{t'} u = \partial_t u + c\partial_x u = 0.$$

This means that u is independent of t', thus the general solution is $u(t,x) = f(-x') = f(x-ct)$.

EXERCISE 2.2.2. Solve $\partial_t u + \partial_x u = 1$.

EXERCISE 2.2.3. Solve $\partial_t u + c\partial_x u + ku = 0$, where c and k are constants.

EXERCISE 2.2.4. Solve $\partial_t u + 2\partial_x u + (2t - x)u = 2t^2 + 3tx - 2x^2$.

We use this example with its explicit solution to bring out some of the notions connected with the *numerical solution* of PDE by the *method of finite differences*. In view of the definition of the partial derivatives, it seems natural to approximate (2.2.2) by the forward difference quotients:

$$\frac{v(t+k,x)-v(t,x)}{k}+c\frac{v(t,x+h)-v(t,x)}{h}=0 \tag{2.2.4}$$

for small positive parameters h,k. We now solve v with initial values $v(0,x)=f(x)$ with a fixed ratio $\lambda=k/h$. We hope that v approximate u as $k\to 0_+$ (iff $h\to 0_+$).

We write (2.2.4) as a recursion formula

$$v(t+k,x)=(1+\lambda c)v(t,x)-\lambda c v(t,x+h).$$

Introducing the *shift operator* $\mathcal{E}$ defined by $\mathcal{E}g(x)=g(x+h)$, we can write the above identity as

$$v(t+k,x)=\left((1+\lambda c)-\lambda c\mathcal{E}\right)v(t,x).$$

Since $(1+\lambda c)$ and $-\lambda c\mathcal{E}$ are commute, by using a formal binomial theorem (which holds true for any operator which are commute), one can easily compute that

$$\begin{aligned}
v(t,x)|_{t=nk} &= v(nk,x)=\left((1+\lambda c)-\lambda c\mathcal{E}\right)^n v(0,x)\\
&=\sum_{m=0}^{n}\binom{n}{m}(1+\lambda c)^m(-\lambda c\mathcal{E})^{n-m}f(x)\\
&=\sum_{m=0}^{n}\binom{n}{m}(1+\lambda c)^m(-\lambda c)^{n-m}f(x+(n-m)h),
\end{aligned}$$

which solves (2.2.4) with initial values $v(x,0)=f(x)$. The domain of dependence for $v(t,x)|_{t=nk}$ is

$$\left\{x,x+h,x+2h,\cdots,x+nh=x+\frac{t}{\lambda}\right\}. \tag{2.2.5}$$

Letting $h,k\to 0_+$ with a fixed ratio $\lambda=k/h$, the limit of the set (2.2.5) (in the topological sense) is the interval $[x,x+(t/\lambda)]$. However, from (2.2.3), the domain of dependence of the solution $u(t,x)$ is $x-ct$, which lies completely outside $[x,x+(t/\lambda)]$. In plain words, this scheme attempts to solve the PDE using some information which is totally irrelevant[1], therefore we do not expect the solution v converges to u as $k\to 0_+$ (iff $h\to 0_+$).

A more appropriate difference scheme uses backward difference quotients:

$$\frac{w(t+k,x)-w(t,x)}{k}+c\frac{w(t,x)-w(t,x-h)}{h}=0. \tag{2.2.6}$$

Using similar computations, under a fixed ratio $\lambda=k/h$ one can show that

$$w(t,x)|_{t=nk}=\sum_{m=0}^{n}\binom{n}{m}(1-\lambda c)^m(\lambda c)^{n-m}f(x-(n-m)h).$$

The domain of dependence for $w(t,x)|_{t=nk}$ is

$$\left\{x,x-h,x-2h,\cdots,x-nh=x-\frac{t}{\lambda}\right\},$$

[1] To be precise, the scheme (2.2.4) does not satisfies the Courant-Friedrichs-Lewy test [**Joh78**].

which is approximating the set $[x, x-(t/\lambda)]$. This scheme looks good, at least now useful information $x-ct$ (see (2.2.3)) is now included in the domain of dependence $[x, x-(t/\lambda)]$ if we choose $\lambda c \leq 1$ (even though it contains some redundant information).

In fact, one can prove that w converges to u as $h \to 0_+$ with the fixed ratio $\lambda = k/h$ with $\lambda c \leq 1$:

$$|u(t,x) - w(t,x)| \leq \frac{Kth}{\lambda} \quad \text{with} \quad K = \frac{1}{2}(c^2\lambda^2 + c\lambda)\|f''\|_{L^\infty(\mathbb{R})}.$$

We will not going to go through all these details, see [**Joh78**] for more details. *In general, given a PDE (with a unique regular solution), the numerical scheme need to be carefully chosen.* One also can refer to the monograph [**AH09**] for detailed explanation about this topic.

Exercise 2.2.5. Using computer software (Octave, MATLAB, SageMath (Python) etc.) to verify the numerical schemes (2.2.4) and (2.2.6). One can choose, for example, f be the bump function, which is C^∞ and has compact support.

2.2.3. The variable coefficient equation. Before we proceed, let us recall some fundamental theorems. We now consider the following standard form of ODE:

$$\gamma'(t) = c(t, \gamma(t)). \tag{2.2.7}$$

We recall the following two fundamental theorems of ODE (see Theorem 2.2.6 and Theorem 2.2.9 below, both of them also hold true for system of equations as well):

Theorem 2.2.6 ([**HS99**, Theorem I-2-5]). *Let $t_0, x_0 \in \mathbb{R}$. Suppose that $(t,x) \mapsto c(t,x)$ is real-valued and continuous on a rectangular region*

$$\mathcal{R} = \{(t,x) : |t-t_0| \leq a, |x-x_0| \leq b\}$$

for some positive numbers a and b. Let

$$\alpha := \begin{cases} a & \text{if } \|c\|_{L^\infty(\mathcal{R})} = 0, \\ \min\{a, b\|c\|_{L^\infty(\mathcal{R})}^{-1}\} & \text{if } \|c\|_{L^\infty(\mathcal{R})} > 0. \end{cases}$$

Then there exists a solution $\gamma \in C^1((t_0-\alpha, t_0+\alpha))$ of the ODE (2.2.7) with $\gamma(t_0) = x_0$.

Example 2.2.7 ([**HS99**, Example I-2-6]). Theorem 2.2.6 applies to the ODE

$$\gamma'(t) = t(\gamma(t))^{\frac{1}{5}}, \quad \gamma(3) = 0. \tag{2.2.8}$$

One sees that $\gamma(t) = 0$ is obviously a solution of the above ODE. In fact, one also verifies that

$$\gamma(t) = \begin{cases} 0 & \text{for } t < 3, \\ \left(\dfrac{2(t^2-9)}{5}\right)^{5/4} & \text{for } t \geq 3, \end{cases} \tag{2.2.9}$$

is also a solution of (2.2.8). This shows that in general the solution of ODE (2.2.7) may not unique.

Exercise 2.2.8. Verify the function γ given in (2.2.9) is $C^1(\mathbb{R})$ and it solves (2.2.8).

We need extra assumptions to guarantee uniqueness.

THEOREM 2.2.9 ([HS99, Theorem I-1-4]). *Suppose that all assumptions in Theorem 2.2.6 hold. We additionally assume that there exists a positive constant L such that*

$$|c(t,x_1) - c(t,x_2)| \leq L|x_1 - x_2| \tag{2.2.10}$$

whenever (t,x_1) and (t,x_2) are in the region $\mathcal{R}$. Then there exists a unique *solution $\gamma \in C^1((t_0-\alpha, t_0+\alpha))$ of the ODE (2.2.7) with $\gamma(t_0) = x_0$.*

We now consider the transport equation with variable coefficient equation of the form

$$\partial_t u + c(t,x)\partial_x u = 0, \quad u(0,x) = f(x), \tag{2.2.11}$$

where $f \in C^1(\mathbb{R})$ and c satisfies all assumption in Theorem 2.2.9. Given any $s \in \mathbb{R}$ and we consider a curve $x = \gamma_s(t)$, where γ solves the ODE (2.2.7) with $\gamma_s(0) = s$. Similar to above, we now restrict u on a curve $x = \gamma_s(t)$, and one sees that

$$\begin{aligned}\partial_t \left(u|_{\gamma_s}(t)\right) &= \partial_t \left(u(t,\gamma_s(t))\right) = (\partial_t u + \gamma'(t)\partial_x u)|_{x=\gamma_s(t)} \\ &= (\partial_t u + c(t,x)\partial_x u)|_{x=\gamma_s(t)} = 0.\end{aligned}$$

This means that u is constant along the characteristic curve γ_s. Hence

$$u(t,\gamma_s(t)) = u(0,\gamma_s(0)) = f(\gamma_s(0)) = f(s). \tag{2.2.12}$$

For later convenience, we write $\gamma(t,s) = \gamma_s(t)$. Fix $x \in \mathbb{R}$ and we now want to solve the equation $x = \gamma(t,s)$. From $\gamma(0,x) = x$, if $\partial_s\gamma(0,x) \neq 0$, then we can apply the implicit function theorem [Apo74, Theorem 13.7] to guarantee that there exist an open neighborhood $U_x \subset \mathbb{R}$ of 0 and $g_x \in C^1(U_x)$ such that $g_x(0) = x$ and $x = \gamma(t,s)|_{s=g_x(t)}$ for all $t \in U_x$. In other words, we found a solution $s = g_x(t) \equiv g(x,t)$ of the equation $x = \gamma(t,s)$ in U_x. Plugging this solution into (2.2.12), we conclude

$$u(t,x) = f(g(x,t)) \quad \text{for all } x \text{ with } \partial_s\gamma(0,x) \neq 0 \text{ and } t \in U_x. \tag{2.2.13}$$

This completes the local existence proof. Uniqueness follows from the fact that u is constant along the characteristic curve γ.

EXAMPLE 2.2.10 (Revisit Section 2.2.2). Given any $f \in C^1(\mathbb{R})$, let us now consider (2.2.11) with $c =$ constant. In this case, (2.2.7) reads $\gamma'(t) = c$. For each $s \in \mathbb{R}$, it is easy to see that the solution of $\gamma_s'(t) = c$ with $\gamma_s(0) = s$ is

$$\gamma(t,s) \equiv \gamma_s(t) = ct + s.$$

For each $x \in \mathbb{R}$, the solution of $x = \gamma(t,s)$ is clearly given by $s = g(x,t) \equiv x - ct$, and thus from (2.2.13) we conclude that

$$u(t,x) = f(x - ct),$$

which agrees with (2.2.3).

EXAMPLE 2.2.11. Given any $f \in C^1(\mathbb{R})$, we now want to solve $\partial_t u + x\partial_x u = 0$ with $u(0,x) = f(x)$ for all $x \in \mathbb{R}$. Write $c(t,x) = x$, and for each $s \in \mathbb{R}$ we consider the ODE

$$\gamma_s'(t) = c(t,\gamma_s(t)) \equiv \gamma_s(t), \quad \gamma_s(0) = s.$$

By using the integrating factor, one can easily see that the solution of the ODE is

$$\gamma_s(t) = e^t s.$$

For each $x \in \mathbb{R}$, the solution of $x = \gamma_s(t)$ is given by $s = g(x,t) \equiv e^{-t}x$, and thus from (2.2.13) we conclude that

$$u(t,x) = f(g(x,t)) = f(e^{-t}x).$$

EXAMPLE 2.2.12. Given any $f \in C^1(\mathbb{R})$, we now want to solve $\partial_t u + 2tx^2 \partial_x u = 0$ with $u(0,x) = f(x)$ for all $x \in \mathbb{R}$. Write $c(t,x) = 2tx^2$, and for each $s \neq 0$ we consider the ODE

$$\gamma_s'(t) = 2t(\gamma_s(t))^2, \quad \gamma_s(0) = s^{-1}.$$

By using the method of separation of variables, one can easily see that the solution of the ODE is

$$\gamma_s(t) = (s - t^2)^{-1},$$

which is valid

$$\begin{cases} \text{for all } t \in \mathbb{R} & \text{when } s < 0, \\ \text{for all } t^2 < s & \text{when } s > 0, \end{cases} \tag{2.2.14}$$

but *the ODE is not solvable* when $s = 0$. When $s \neq 0$, the solution of $x = \gamma_s(t)$ is given by $s = t^2 + \frac{1}{x}$, and thus from (2.2.13) we conclude that

$$u(t,x) = f(s^{-1}) = f\left(\frac{x}{1 + t^2 x}\right) \quad \text{for all } x \neq -t^{-2}.$$

REMARK 2.2.13. Recall that the local existence proof for the solution of the PDE (2.2.11) involving implicit function theorem, which requires $c(0,x) \neq 0$. As we see in Example 2.2.10 and Example 2.2.11, the PDE also solvable for those x with $c(0,x) = 0$. However, in Example 2.2.12, we see that the PDE is not solvable for those x with $c(0,x) = 0$. The existence theorem guarantees the local solvable when x with $c(0,x) \neq 0$ and inconclusive when $c(0,x) = 0$: The implicit function theorem provided only *sufficient condition*, but not *necessary condition*. In addition, the existence theorem does not guarantee the maximal domain, the solution may global (i.e. valid for all $t \in \mathbb{R}$) in some case.

We now summarize the above ideas in the following algorithm:

Algorithm 1 Solving $\partial_t u + c(t,x)\partial_x u + d(t,x)u = F(t,x)$ with $u(0,x) = f(x)$

1: Solve the ODE $\gamma_s'(t) = c(t, \gamma_s(t))$ with given $\gamma_s(0)$ for any suitable parameter s.
2: Compute $\partial_t(u(t, \gamma_s(t)))$.
3: Rewrite the identity $x = \gamma_s(t)$ in the form of $s = g(x,t)$.
4: Identify the domain for which $u(t,x) = f(g(x,t))$ solves $\partial_t u + c(t,x)\partial_x u = 0$.

EXERCISE 2.2.14. Given any $f \in C^1(\mathbb{R})$, solve the equation $(1+t^2)\partial_t u + \partial_x u = 0$ with $u(0,x) = f(x)$ and identify the range of x.

EXERCISE 2.2.15. Given any $f \in C^1(\mathbb{R})$, solve the equation $t\partial_t u + x\partial_x u = 0$ with $u(0,x) = f(x)$ and identify the range of x.

EXERCISE 2.2.16. Solve the equation $x\partial_t u + t\partial_x u = 0$ with $u(0,x) = e^{-x^2}$.

REMARK (Generality of the ideas). The method of characteristic also works for fairly general Cauchy problem of the (high dimensional) first order inhomogeneous quasilinear equation (even works for some cases for fully nonlinear case), see [Joh78].

2.3. Linear PDE of second order

We now move on to the linear second order PDE. We usually classify them by considering its principal part (i.e. the term with highest order derivatives). For simplicity, let us consider the principal part of the constant coefficient case:

$$A : \nabla^{\otimes 2} u := \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j u.$$

Suppose that the matrix $A = (a_{ij})$ is real symmetric. It is well-known that A is unitary diagonalizable, i.e. there exists an invertible Q with $Q^{-1} = Q^{\intercal}$ such that $A = QDQ^{\intercal}$ for some diagonal matrix $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, where λ_j are called the *eigenvalues* of the matrix A.

(1) If $\lambda_j > 0$ for all $j = 1, \cdots, n$, then we say that second order linear PDE is *elliptic*.
(2) If $\lambda_{j_0} = 0$ and $\lambda_j > 0$ for all $j \in \{1, \cdots, n\} \setminus \{j_0\}$, then we say that the second order linear PDE is *parabolic*.
(3) If $\lambda_{j_0} < 0$ and $\lambda_j > 0$ for all $j \in \{1, \cdots, n\} \setminus \{j_0\}$, then we say that the second order linear PDE is *hyperbolic*.
(4) If $n \geq 4$ and there exists 4 different indices $j_0, j_0', j_1, j_1' \in \{1, \cdots, n\}$ such that $\lambda_{j_0} < 0$, $\lambda_{j_0'} < 0$, $\lambda_{j_1} > 0$ and $\lambda_{j_1'} > 0$, then we say that the second order linear PDE is *ultrahyperbolic*.

We have to emphasize that the above are not complete classifications of the second order linear PDE.

Exercise 2.3.1. Let $A = (a_{ij})$ be a real symmetric matrix. Show that all its eigenvalue are positive if and only if

$$A\boldsymbol{\xi} \cdot \boldsymbol{\xi} \equiv \boldsymbol{\xi}^{\intercal} A \boldsymbol{\xi} > 0 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n.$$

Remark 2.3.2. If we consider the linear second order PDE with principal part

$$A(\boldsymbol{x}) : \nabla^{\otimes 2} u \equiv \sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) \partial_i \partial_j u, \quad \text{or} \tag{2.3.1a}$$

$$\nabla \cdot (A(\boldsymbol{x}) \nabla u) \equiv \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(\boldsymbol{x}) \partial_j u \right), \tag{2.3.1b}$$

then we say that (2.3.1a) and (2.3.1b) is *uniformly elliptic* on a domain Ω if there exists a constant $c > 0$ such that

$$A(\boldsymbol{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \equiv \boldsymbol{\xi}^{\intercal} A(\boldsymbol{x}) \boldsymbol{\xi} \geq c|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{x} \in \Omega \text{ and } \boldsymbol{\xi} \in \mathbb{R}^n.$$

We usually call (2.3.1a) the second order elliptic operator of *non-divergence form*, while (2.3.1b) the second order elliptic operator of *divergence form*. In many cases, it is more convenient to consider the divergence form (2.3.1b) since it is symmetric with respect to the integration by parts formula Corollary 1.0.19. In particular when $A \equiv \mathrm{Id}$, then both (2.3.1a) and (2.3.1b) reduce into the *Laplace operator* (or *Laplacian*):

$$\Delta = \sum_{i=1}^{n} \partial_i^2 u.$$

Exercise 2.3.3. Let $n = 2$. Classify each of the equations:

(1) $\partial_1^2 u - 5\partial_1\partial_2 u = 0$;
(2) $4\partial_1^2 u - 12\partial_1\partial_2 u + 9\partial_2^2 u + \partial_2 u = 0$;
(3) $4\partial_1^2 u + 6\partial_1\partial_2 u + 9\partial_2^2 u = 0$.

EXERCISE 2.3.4 (Standard examples of second order linear PDE). Classify each of the equations:

(1) $-\Delta u + \boldsymbol{b}\cdot\nabla u + cu = 0$ with $u = u(\boldsymbol{x})$, where $\nabla = (\partial_1, \cdots, \partial_n)$ is the gradient operator.
(2) **Heat equation (or caloric equation, diffusion equation).** $\partial_t u - \Delta u + \boldsymbol{b}\cdot\nabla u + cu = 0$ with $u = u(t,\boldsymbol{x})$.
(3) **Wave equation.** $\partial_t^2 u - \Delta u + \boldsymbol{b}\cdot\nabla u + cu = 0$ with $u = u(t,\boldsymbol{x})$.
(4) Suppose that $\mathcal{L}u = A : \nabla^{\otimes 2} u$, where $A = (a_{ij})$ be a real symmetric positive definite matrix (in the sense of Exercise 2.3.1). What can we say about the operator $\partial_t - \mathcal{L}$ as well as $\partial_t^2 - \mathcal{L}$?

REMARK 2.3.5. Many authors (including myself) would prefer put a minus sign in front of the Laplacian Δ (or the elliptic operator (2.3.1b)) as indicated in Exercise 2.3.4, due to the maximum principle (Lemma 3.5.5), eigenvalue decomposition (Theorem 3.6.4) as well as Fourier transform (Exercise 4.2.8) below. This minus sign is actually come from the integration by parts (Corollary 1.0.19).

2.4. Wave equation

2.4.1. 1-dimensional wave equation on the whole line $\mathbb{R}$. We now begin our discussions by studying the C^2-solution of the one-dimensional wave equation

$$\partial_t^2 u(t,x) = c^2\partial_x^2 u(t,x) \quad \text{for } (t,x)\in\mathbb{R}\times\mathbb{R}, \tag{2.4.1}$$

for some $c > 0$, which is also the simplest form of second order *hyperbolic* PDE. The above wave equation can be written as (because the differential operators are linear and commute)

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0 \quad \text{for } (t,x)\in\mathbb{R}\times\mathbb{R}, \tag{2.4.2}$$

in other words, the function $v := (\partial_t + c\partial_x)u \equiv \partial_t u + c\partial_x u$ satisfies the transport equation. One may use the above method to solve the general solution of (2.4.1). Here we exhibit another method to solve (2.4.1) using *characteristic coordinate* given by

$$\xi = x + ct, \quad \eta = x - ct.$$

For simplicity, here we slightly abuse the notation by identifying $v(t,x)$ with $v(\xi,\eta)$. By using chain rule, one sees that $\partial_x v = \partial_\xi v + \partial_\eta v$ and $\partial_t v = c\partial_\xi v + c\partial_\eta v$. Since these two identity holds true for arbitrary $v\in C^1$, we simply write

$$\partial_x = \partial_\xi + \partial_\eta, \quad \partial_t = c\partial_\xi + c\partial_\eta.$$

In view of (2.4.2), we write

$$\partial_t - c\partial_x = -2c\partial_\eta, \quad \partial_t + c\partial_x = 2c\partial_\xi$$

and hence this change of coordinate transform (2.4.2) in the simple form

$$\partial_\eta\partial_\xi u = 0.$$

Then obviously the general solution is $u(\xi,\eta) = f(\xi) + g(\eta)$ with $f,g\in C^2(\mathbb{R})$. In terms of original coordinate, we reach

$$u(t,x) = f(x+ct) + g(x-ct) \quad \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{2.4.3}$$

Since there are two "unknowns" f and g in (2.4.3), to guarantee the existence and uniqueness of solution of the PDE, one way is to impose the following initial conditions:

$$u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x), \tag{2.4.4}$$

for any given $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$.

Choosing $t = 0$ in (2.4.3) yields

$$\phi(x) = f(x) + g(x),$$

and hence

$$\phi'(x) = f'(x) + g'(x). \tag{2.4.5a}$$

On the other hand, we first differentiate (2.4.3) and then take $t = 0$ to obtain

$$\psi(x) = cf'(x) - cg'(x). \tag{2.4.5b}$$

From (2.4.5a) and (2.4.5b), it is not difficult to see that (for later convenience, we replace x by s)

$$f'(s) = \frac{1}{2}\phi'(s) + \frac{1}{2c}\psi(s), \quad g'(s) = \frac{1}{2}\phi'(s) - \frac{1}{2c}\psi(s),$$

and thus integrating $\mathrm{d}x$ from 0 to $y \in \mathbb{R}$, we reach

$$\begin{aligned} f(y) - f(0) &= \frac{1}{2}\phi(y) - \frac{1}{2}\phi(0) + \frac{1}{2c}\int_0^y \psi(s)\,\mathrm{d}s, \\ g(y) - g(0) &= \frac{1}{2}\phi(y) - \frac{1}{2}\phi(0) - \frac{1}{2c}\int_0^y \psi(s)\,\mathrm{d}s. \end{aligned}$$

Plugging these anzats into the general solution (2.4.3), by using the fact $\phi(0) = f(0) + g(0)$ we reach the *d'Alembert formula*:

$$u(t,x) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s. \tag{2.4.6}$$

Until this point, we have showed that:

LEMMA. *Let $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Let $u \in C^2(\mathbb{R} \times \mathbb{R})$ be a solution of the initial-value problem of the 1-dimensional wave equation (2.4.1) and (2.4.4), then the solution must takes the form (2.4.6).*

If $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$, one can directly verify that (2.4.6) solves the initial-value problem of the 1-dimensional wave equation (2.4.1) and (2.4.4), which shows the existence of the solution. Putting the above together, we now conclude that:

THEOREM 2.4.1 (d'Alembert). *Let $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Then the function (2.4.6) is the unique C^2-solution of the initial-value problem of the 1-dimensional wave equation (2.4.1) and (2.4.4).*

It is interesting to mention that, despite that the wave equation (2.4.1) involving second order derivatives, and the initial condition (2.4.4) involving first order derivatives, but the d'Alembert formula formula (2.4.6) is actually well-defined without diffrentiability assumption. In fact, in many practical situation (see Exercise 2.4.2 and Exercise 2.4.3 below), we usually do not expect the wave to be C^2. We will introduce in next chapter that the notion of *weak solutions* in terms of *weak derivatives*.

EXERCISE 2.4.2 (The plucked string). The solution of the 1-dimensional wave equation (2.4.1) can be used to approximate the pointwise displacement of the vibrating of an (infinitely) long string. The sound speed c is given by $c = \sqrt{T/\rho}$, where T is the tension of string and ρ is the density of the string. Consider the string with initial position

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases}$$

and initial velocity $\psi(x) \equiv 0$. This modeling the "three-finger" pluck, with all three fingers removed at once. Note that $\phi \in C^0(\mathbb{R})$ but not differentiable at $x = 0, \pm a$. Compute the solution u from the d'Alembert formula (2.4.6). [Hint: Consider the cases $t < \frac{a}{c}$, $t = \frac{a}{c}$, $t > \frac{a}{c}$]

EXERCISE 2.4.3 (The hammer blow). Let $\phi(x) \equiv 0$ and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a. \end{cases}$$

In this case, ψ is not differentiable at $x = \pm a$. Compute the solution u from the d'Alembert formula (2.4.6).

Let u be any C^2 solution of the 1-dimensional wave equation (2.4.1), such that

$$\operatorname{supp}(u(t,\cdot)) := \overline{\{x \in \mathbb{R} : u(t,x) \neq 0\}} \tag{2.4.7}$$

is compact for each t. In view of the d'Alembert's formula (Theorem 2.4.1), this assumption make sense. Recall that $c = \sqrt{T/\rho}$, where T is the tension of string and ρ is the density of the string. In this case, the quantity

$$E(t) := \frac{1}{2} \int_{-\infty}^{\infty} \left(\rho|\partial_t u(t,x)|^2 + T|\partial_x u(t,x)|^2\right) \mathrm{d}x \tag{2.4.8}$$

is well-defined. We have the following lemma (which is far away from optimal).

LEMMA 2.4.4 (see e.g. [**Str08**, Appendix A.3]). *Let $f(t,x)$ and $\partial_t f(t,x)$ be continuous functions in $(c,d) \times \mathbb{R}$. Assume that*

$$\sup_{t \in (c,d)} \|f(t,\cdot)\|_{L^1(\mathbb{R})} \equiv \sup_{t \in (c,d)} \int_{-\infty}^{+\infty} |f(t,x)| \,\mathrm{d}x < \infty,$$

$$\sup_{t \in (c,d)} \|\partial_t f(t,\cdot)\|_{L^1(\mathbb{R})} \equiv \sup_{t \in (c,d)} \int_{-\infty}^{+\infty} |\partial_t f(t,x)| \,\mathrm{d}x < \infty.$$

Then

$$\partial_t \left(\int_{-\infty}^{+\infty} f(t,x) \,\mathrm{d}x \right) = \int_{-\infty}^{+\infty} \partial_t f(t,x) \,\mathrm{d}x.$$

SKETCH OF PROOF. For each $t \in (c,d)$, one can find $c < c' < d' < d$ such that $t \in (c', d')$. This is usual trick in real analysis since in general supremum/infimum cannot be achieved for original large domain. For each $\tau \neq 0$ with $t + \tau \in (c', d')$, which can be achieved by small $|\tau|$, it is easy to see that

$$\frac{1}{\tau} \left(\int_{-\infty}^{+\infty} f(t+\tau, x) \,\mathrm{d}x - \int_{-\infty}^{+\infty} f(t,x) \,\mathrm{d}x \right) = \int_{-\infty}^{+\infty} \frac{1}{\tau} \left(f(t+\tau, x) - f(t,x) \right) \mathrm{d}x.$$

The idea is to take $\tau \to 0$, which can be rigorously justify using the Lebesgue dominated convergence theorem (Lemma 1.0.2), here we will skip the details. $\square$

We differentiate (2.4.8) to obtain

$$\begin{aligned}
E'(t) &:= \frac{1}{2}\int_{-\infty}^{\infty} \left(\rho\partial_t\left(|\partial_t u(t,x)|^2\right) + T\partial_t\left(|\partial_x u(t,x)|^2\right)\right) \mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\rho\partial_t u(t,x)\partial_t^2 u(t,x) + T\partial_x\partial_t u(t,x)\partial_x u(t,x)\right) \mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\rho\partial_t u(t,x)\partial_t^2 u(t,x) - T\partial_t u(t,x)\partial_x^2 u(t,x)\right) \mathrm{d}x = 0,
\end{aligned}$$

where the last identity follows from the integration by parts on the variable x together with the support condition (2.4.7). This means that $E \equiv$ constant, and we usually refer the quantity E given in (2.4.8) the "energy" (in view of the energy conservation). Since

$$E_{\mathrm{kinetic}}(t) := \frac{1}{2}\int_{-\infty}^{\infty} \rho|\partial_t u(t,x)|^2 \,\mathrm{d}x$$

is the kinetic energy of the wave particles, then its potential energy can be *approximate* by

$$E_{\mathrm{potential}}(t) := \frac{1}{2}\int_{-\infty}^{\infty} T|\partial_x u(t,x)|^2 \,\mathrm{d}x.$$

This argument actually works in a fairly general framework, see e.g. [**KK22**, Appendix B].

2.4.2. 1-dimensional wave equation on the half-line $(0,\infty)$. We now consider the following initial-boundary value problem:

$$\begin{cases} \partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty)\times(0,\infty), \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x \in (0,\infty), \\ u(t,0) = 0 & \text{for all } t \in (0,\infty), \end{cases} \tag{2.4.9}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ with *compatibility condition* $\phi(0) = \phi'(0) = \phi''(0) = \psi(0) = \psi'(0) = 0$. Here we restrict ourselves when $t > 0$, as the case $t < 0$ can be obtained by the change of variable $t \mapsto -t$. This modeling the vibrating (infinitely) long string with one end fixed. We now consider the *odd extensions* of ϕ and ψ defined by

$$\phi_{\mathrm{odd}}(x) := \begin{cases} \phi(x) & \text{for } x \geq 0, \\ -\phi(-x) & \text{for } x < 0, \end{cases} \quad \psi_{\mathrm{odd}}(x) := \begin{cases} \psi(x) & \text{for } x \geq 0, \\ -\psi(-x) & \text{for } x < 0, \end{cases} \tag{2.4.10}$$

equivalently, $\phi_{\mathrm{odd}}(x) = \mathrm{sign}\,(x)\phi(|x|)$ and similar formula holds for ψ_{odd}. The compatibility conditions guarantee that $\phi_{\mathrm{odd}} \in C^2(\mathbb{R})$ and $\psi_{\mathrm{odd}} \in C^1(\mathbb{R})$, therefore the d'Alembert function (2.4.6) given by

$$u(t,x) := \frac{1}{2}\left(\phi_{\mathrm{odd}}(x+ct) + \phi_{\mathrm{odd}}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\mathrm{odd}}(s)\,\mathrm{d}s \tag{2.4.11}$$

is well-defined and satisfies (2.4.9), which completes the proof of *existence* result (we do not show *uniqueness*).

EXERCISE 2.4.5. Show that (2.4.11) can be written as

$$u(t,x) = \begin{cases} \frac{1}{2}\left(\phi(x+ct)+\phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,\mathrm{d}s & \text{for } x \geq ct, \\ \frac{1}{2}\left(\phi(x+ct)-\phi(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{ct+x}\psi(s)\,\mathrm{d}s & \text{for } 0 < x < ct. \end{cases}$$

Similar as before, the function in Exercise 2.4.5 is well-defined without assuming the differentiability of ϕ and ψ, and even well-defined without the compatibility condition $\phi(0) = \psi(0) = 0$. One can easily check that $\lim_{x\to 0_+} u(t,x) = 0$ for all $t > 0$ (at least) when ϕ and ψ are continuous. Therefore the solution in Exercise 2.4.5 can be interpreted as a *weak solution* of the initial-boundary value problem (2.4.9).

EXERCISE 2.4.6 (Neumann boundary condition). Find a solution of the initial-boundary value problem

$$\begin{cases} \partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = 0 & \text{for all } (t,x)\in(0,\infty)\times(0,\infty), \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x\in(0,\infty), \\ \partial_x u(t,0) = 0 & \text{for all } t\in(0,\infty), \end{cases}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ with *compatibility condition* $\phi(0) = \phi'(0) = \phi''(0) = \psi(0) = \psi'(0) = 0$.

We now consider the following initial-boundary value problem:

$$\begin{cases} \partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = 0 & \text{for all } (t,x)\in(0,\infty)\times(0,\infty), \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x\in(0,\infty), \\ u(t,0) = h(t) & \text{for all } t\in(0,\infty), \end{cases} \tag{2.4.12}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ as well as $h \in C^2((0,\infty))$. By consider

$$\begin{cases} \partial_t^2 v(t,x) - c^2\partial_x^2 v(t,x) = 0 & \text{for all } (t,x)\in(0,\infty)\times(0,\infty), \\ v(0,x) = \phi(x), \quad \partial_t v(0,x) = \psi(x) & \text{for all } x\in(0,\infty), \\ v(t,0) = 0 & \text{for all } t\in(0,\infty), \end{cases}$$

which was discussed above (which can be done without assuming the compatibility conditions), we see that the $w = u - v$ solves

$$\begin{cases} \partial_t^2 w(t,x) - c^2\partial_x^2 w(t,x) = 0 & \text{for all } (t,x)\in(0,\infty)\times(0,\infty), \\ w(0,x) = \partial_t w(0,x) = 0 & \text{for all } x\in(0,\infty), \\ w(t,0) = h(t) & \text{for all } t\in(0,\infty). \end{cases}$$

It remains to solve w. If we have the compatibility conditions $h(0) = h'(0) = h''(0) = 0$, one sees that

$$w(t,x) = \chi_{\{x<ct\}} h\left(t - \frac{x}{c}\right)$$

is our desired C^2-solution. Hence we see that

$$u(t,x) := v(t,x) + \chi_{\{x<ct\}} h\left(t - \frac{x}{c}\right)$$

is a solution of (2.4.12). Again, the above expression is well-defined without the differentiability of h, as well as the compatibility conditions. This again induces a weak solution of (2.4.12).

EXERCISE 2.4.7. Find a weak solution of the initial-boundary value problem

$$\begin{cases} \partial_t^2 u(t,x) - c\partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty)\times(0,\infty), \\ u(0,x) = 0, \quad \partial_t u(0,x) = V & \text{for all } x \in (0,\infty), \\ \partial_t u(t,0) + a\partial_x u(t,0) = 0 & \text{for all } t \in (0,\infty), \end{cases}$$

where V, a, c are positive constants with $a > c$.

2.4.3. 1-dimensional wave equation on the finite interval $(0, L)$. We now consider the following initial-boundary value problem:

$$\begin{cases} \partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty)\times(0,L), \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x \in (0,L), \\ u(t,0) = u(t,L) = 0 & \text{for all } t \in (0,\infty), \end{cases} \tag{2.4.13}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ with compatibility conditions

$$\phi(0) = \phi'(0) = \phi''(0) = \psi(0) = \psi'(0) = 0,$$
$$\phi(L) = \phi'(L) = \phi''(L) = \psi(L) = \psi'(L) = 0.$$

In this case, we consider the odd and periodic extension of ϕ as follows:

$$\phi_{\mathrm{ext}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < L, \\ -\phi(-x) & \text{for } -L < x < 0, \\ \phi_{\mathrm{ext}}(x + 2kL) & \text{for all } x \in \mathbb{R} \text{ and } k \in \mathbb{Z}, \end{cases}$$

and similar extension ψ_{ext} of ψ also defined.

EXERCISE 2.4.8. Show that $\phi_{\mathrm{ext}}(x) = \phi_{\mathrm{odd}}(x - 2L\lfloor x/2L \rfloor)$ and

$$\phi_{\mathrm{ext}}(x) = \begin{cases} \phi(x - \lfloor x/L \rfloor L) & \text{if } \lfloor x/L \rfloor \text{ is even}, \\ -\phi(x - \lfloor x/L \rfloor L - L) & \text{if } \lfloor x/L \rfloor \text{ is odd}. \end{cases}$$

Of course, similar formula holds for ψ.

The compatibility conditions guarantee that $\phi_{\mathrm{odd}} \in C^2(\mathbb{R})$ and $\psi_{\mathrm{odd}} \in C^1(\mathbb{R})$, therefore the d'Alembert function (2.4.6) given by

$$u(t,x) := \frac{1}{2}\left(\phi_{\mathrm{ext}}(x+ct) + \phi_{\mathrm{ext}}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\mathrm{ext}}(s)\,\mathrm{d}s \tag{2.4.14}$$

is well-defined and satisfies (2.4.13), which completes the proof of *existence* result. Similarly, one can consider the weak solution without assuming the differentiability of ϕ and ψ, as well as the compatibility conditions. The solution can be written down explicitly [**Str08**], and hence we will not going to elaborate these computations here. In fact, the (weak) solution of the initial-boundary value problem (2.4.13) is unique. We will explain this in the next chapter.

EXERCISE 2.4.9 (Neumann boundary condition). Find a solution of the initial-boundary value problem

$$\begin{cases} \partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty)\times(0,L), \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x \in (0,L), \\ \partial_x u(t,0) = \partial_x u(t,L) = 0 & \text{for all } t \in (0,\infty), \end{cases}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ with compatibility conditions

$$\begin{aligned}\phi(0) = \phi'(0) = \phi''(0) = \psi(0) = \psi'(0) = 0,\\ \phi(L) = \phi'(L) = \phi''(L) = \psi(L) = \psi'(L) = 0.\end{aligned}$$

Verify the conditions explicitly.

EXERCISE 2.4.10. Find a solution of the initial-boundary value problem

$$\begin{cases}\partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty)\times(0,L),\\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x \in (0,L),\\ u(t,0) = 0, \quad \partial_x u(t,L) = 0 & \text{for all } t \in (0,\infty),\end{cases}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ with compatibility conditions

$$\begin{aligned}\phi(0) = \phi'(0) = \phi''(0) = \psi(0) = \psi'(0) = 0,\\ \phi(L) = \phi'(L) = \phi''(L) = \psi(L) = \psi'(L) = 0.\end{aligned}$$

Verify the conditions explicitly.

2.4.4. 1-dimensional wave with an external source: Duhamel's principle. We now solve

$$\begin{cases}\partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = f(t,x) & \text{for } (t,x) \in (0,\infty)\times\mathbb{R},\\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for } x \in \mathbb{R},\end{cases} \tag{2.4.15}$$

where $f \in C^1(\mathbb{R}^2)$, $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Let v be a C^2-solution of the homogeneous problem

$$\begin{cases}\partial_t^2 v(t,x) - c^2\partial_x^2 v(t,x) = 0 & \text{for } (t,x) \in (0,\infty)\times\mathbb{R},\\ v(0,x) = \phi(x), \quad \partial_t v(0,x) = \psi(x) & \text{for } x \in \mathbb{R},\end{cases}$$

then we see that the function $w := u - v$ satisfies

$$\begin{cases}\partial_t^2 w(t,x) - c^2\partial_x^2 w(t,x) = f(t,x) & \text{for } (t,x) \in (0,\infty)\times\mathbb{R},\\ w(0,x) = \partial_t w(0,x) = 0 & \text{for } x \in \mathbb{R}.\end{cases} \tag{2.4.16}$$

The remaining task is to find a C^2-solution w.

There are several ways to compute w. Here we will use Duhamel's principle to reduce the *inhomogeneous* initial-value problem (2.4.16) into a *homogeneous* one. The reason we exhibit this idea since it can be easily extended for *higher order hyperbolic equations with constant coefficients*. To clarify the ideas, here we only consider wave equation, see [**Joh78**] for more details.

For each auxiliary parameter $\tau \geq 0$, we consider the homogeneous initial-value problem

$$\begin{cases}\partial_t^2 W(t,x;\tau) - c^2\partial_x^2 W(t,x;\tau) = 0 & \text{for all } t \geq \tau \text{ and } x \in \mathbb{R},\\ W(\tau,x,;\tau) = 0, \quad \partial_t W(\tau,x;\tau) = f(\tau,x) & \text{for all } x \in \mathbb{R}.\end{cases}$$

Then we can solve the above equation via d'Alembert formula (2.4.6):

$$W(t,x;\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,s)\,\mathrm{d}s.$$

We now define

$$w(t,x) := \int_0^t W(t,x;\tau)\,\mathrm{d}\tau.$$

One can check that (the interchangability of integral and ∂_t need to be justified)

$$\partial_t w(t,x) = \overbrace{W(t,x;t)}^{=0} + \int_0^t \partial_t W(t,x;\tau)\,\mathrm{d}\tau = \int_0^t \partial_t W(t,x;\tau)\,\mathrm{d}\tau,$$

$$\partial_t^2 w(t,x) = \overbrace{\partial_t W(t,x;t)}^{=f(t,x)} + \overbrace{\int_0^t \partial_t^2 W(t,x;\tau)\,\mathrm{d}\tau}^{=c^2\int_0^t \partial_x^2 W(t,x;\tau)\,\mathrm{d}\tau} = f(t,x) + c^2\partial_x^2 w(t,x),$$

which means that

$$w(t,x) := \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau$$

is a solution of the inhomogeneous problem (2.4.16). Therefore, we found a C^2-solution of the inhomogeneous problem (2.4.15) given by the d'Alembert formula:

$$u(t,x) = \frac{1}{2}\left(\phi(x+ct)+\phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,\mathrm{d}s + \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau.$$

By considering the method of extensions, one also can solve the problem on half-space $(0,\infty)$ as well as on the finite interval $(0,L)$.

2.4.5. n-dimensional wave equation in space time. We now consider the n-dimensional Laplace operator $\Delta = \partial_1^2 + \partial_2^2 + \cdots + \partial_n^2$ in spatial variables. We are looking for an (explicit) solution of

$$\begin{cases} \partial_t^2 u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = 0 & \text{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x}\in\mathbb{R}^n, \end{cases} \tag{2.4.17}$$

where $\phi \in C^3(\mathbb{R}^n)$ and $\psi \in C^2(\mathbb{R}^n)$ (here we impose slightly higher regularity than the one-dimensional case).

Exercise 2.4.11 (Conservation of energy). Assume some suitable support conditions, define an energy functional for the n-dimensional wave equation $\partial_t^2 u - c\Delta u = 0$ in $\mathbb{R}^n$, similar to (2.4.8), and show that such energy is conservative.

For each $h \in C^2(\mathbb{R}^n)$, we define an auxiliary function

$$M_h(\boldsymbol{x},r) := \frac{1}{\mathscr{H}^{n-1}(\partial B_r(\boldsymbol{x}))}\int_{\partial B_r(\boldsymbol{x})} h(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}} \quad \text{for all } r>0, \tag{2.4.18}$$

where $\mathscr{H}^{n-1}$ is the Hausdorff measure. By using a simple change of variable, one sees that

$$M_h(\boldsymbol{x},r) = \frac{1}{\omega_n}\int_{\mathcal{S}^{n-1}} h(\boldsymbol{x}+r\hat{\boldsymbol{z}})\,\mathrm{d}\hat{\boldsymbol{z}}, \tag{2.4.19}$$

where $\mathcal{S}^{n-1} := \partial B_1(0)$ and $\omega_n := \mathscr{H}^{n-1}(\mathcal{S}^{n-1})$. It is important to notice that (2.4.19) is well-defined for all $r\in\mathbb{R}$, and it is even with respect to r, i.e. $M_h(\boldsymbol{x},r) = M_h(\boldsymbol{x},-r)$ for all $r\in\mathbb{R}$.

In this case, $\mathscr{H}^{n-1}(\partial B_r(\boldsymbol{x}))$ is just simply the surface area of $\partial B_r(\boldsymbol{x})$, and its formula is given by

$$\mathscr{H}^{n-1}(\partial B_r(\boldsymbol{x})) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1},$$

where Γ is the usual gamma function. By dominated convergence theorem (Lemma 1.0.2), one sees that the differential operator is commute with the integral, and we see that $M_h \in C^2(\mathbb{R}^{n+1})$. In particular, we compute that

$$\begin{aligned}
\partial_r M_h(\boldsymbol{x}, r) &= \frac{1}{\omega_n} \int_{\mathcal{S}^{n-1}} \partial_r \left(h(\boldsymbol{x} + r\hat{\boldsymbol{z}}) \right) \mathrm{d}\hat{\boldsymbol{z}} \\
&= \frac{1}{\omega_n} \int_{\mathcal{S}^{n-1}} \nabla h(\boldsymbol{y})|_{\boldsymbol{y}=\boldsymbol{x}+r\hat{\boldsymbol{z}}} \cdot \hat{\boldsymbol{z}} \, \mathrm{d}\hat{\boldsymbol{z}} \quad \text{(chain rule)} \\
&= \frac{1}{\omega_n r} \int_{\mathcal{S}^{n-1}} \hat{\boldsymbol{z}} \cdot \nabla_{\boldsymbol{z}} \left(h(\boldsymbol{x} + r\boldsymbol{z}) \right)|_{\boldsymbol{z}=\hat{\boldsymbol{z}}} \, \mathrm{d}\hat{\boldsymbol{z}}
\end{aligned}$$

Then using divergence theorem, we further compute that[2]

$$\begin{aligned}
\partial_r M_h(\boldsymbol{x}, r) &= \frac{1}{\omega_n r} \int_{B_1(0)} \Delta_{\boldsymbol{z}} \left(h(\boldsymbol{x} + r\boldsymbol{z}) \right) \mathrm{d}\boldsymbol{z} \quad \text{(divergence theorem)} \\
&= \frac{r}{\omega_n} \int_{B_1(0)} \Delta_{\boldsymbol{x}} h(\boldsymbol{x} + r\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z} \quad \text{(chain rule)} \\
&= \frac{r}{\omega_n} \Delta_{\boldsymbol{x}} \left(\int_{B_1(0)} h(\boldsymbol{x} + r\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z} \right) \\
&= \frac{r^{1-n}}{\omega_n} \Delta_{\boldsymbol{x}} \left(\int_{B_r(\boldsymbol{x})} h(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right) \\
&= \frac{r^{1-n}}{\omega_n} \Delta_{\boldsymbol{x}} \left(\int_0^r \int_{\partial B_\rho(\boldsymbol{x})} h(\boldsymbol{y}) \, \mathrm{d}S_{\boldsymbol{y}} \, \mathrm{d}\rho \right) \quad \text{(polar coordinate)} \\
&= r^{1-n} \Delta_{\boldsymbol{x}} \left(\int_0^r \rho^{n-1} M_h(\boldsymbol{x}, \rho) \, \mathrm{d}\rho \right).
\end{aligned}$$

Therefore, we reach

$$\partial_r \left(r^{n-1} \partial_r M_h(\boldsymbol{x}, r) \right) = \Delta_{\boldsymbol{x}} \partial_r \left(\int_0^r \rho^{n-1} M_h(\boldsymbol{x}, \rho) \, \mathrm{d}\rho \right) = \Delta_{\boldsymbol{x}} r^{n-1} M_h(\boldsymbol{x}, r).$$

Hence we reach the *Darboux's equation*:

$$r^{1-n} \partial_r \left(r^{n-1} \partial_r M_h(\boldsymbol{x}, r) \right) = \Delta_{\boldsymbol{x}} M_h(\boldsymbol{x}, r), \tag{2.4.20}$$

with "initial" conditions

$$M_h(\boldsymbol{x}, 0) = h(\boldsymbol{x}) \quad \text{(by definition of } M_h \text{ and mean value theorem)} \tag{2.4.21a}$$

$$\partial_r M_h(\boldsymbol{x}, r)|_{r=0} = \frac{r}{\omega_n} \int_{B_1(0)} \overbrace{\Delta_{\boldsymbol{x}} h(\boldsymbol{x} + r\boldsymbol{z})}^{\text{uniformly bounded}} \, \mathrm{d}\boldsymbol{z} \Bigg|_{r \to 0} = 0 \tag{2.4.21b}$$

[2]Polar coordinate is a very particular case of coarea formula [Cha06].

EXERCISE 2.4.12. Suppose that $g \in C^2$ is radially symmetric, i.e. $g(\boldsymbol{y}) = g(r)$ with $r = |\boldsymbol{y}|$. Show that

$$\Delta_{\boldsymbol{y}} g = r^{1-n}\partial_r\left(r^{n-1}\partial_r g\right) = \partial_r^2 g + \frac{n-1}{r}\partial_r g.$$

We now choose $h = u(t,\cdot)$ and we write

$$M_u(t,\boldsymbol{x},r) := M_{u(t,\cdot)}(\boldsymbol{x},r) = \frac{1}{\omega_n}\int_{\mathcal{S}^{n-1}} u(t,\boldsymbol{x}+r\hat{\boldsymbol{z}})\,\mathrm{d}\hat{\boldsymbol{z}}.$$

From (2.4.21a), we have

$$M_u(t,\boldsymbol{x},0) = u(t,\boldsymbol{x}).$$

On the other hand, by using dominated convergence theorem (Lemma 1.0.2), one can verify that

$$\begin{aligned}\Delta_{\boldsymbol{x}} M_u(t,\boldsymbol{x},r) &= \frac{1}{\omega_n}\int_{\mathcal{S}^{n-1}} \Delta_{\boldsymbol{x}} u(t,\boldsymbol{x}+r\hat{\boldsymbol{z}})\,\mathrm{d}\hat{\boldsymbol{z}} \\ &= \frac{1}{c^2}\partial_t^2\frac{1}{\omega_n}\int_{\mathcal{S}^{n-1}} u(t,\boldsymbol{x}+r\hat{\boldsymbol{z}})\,\mathrm{d}\hat{\boldsymbol{z}} = \frac{1}{c^2}\partial_t^2 M_u(t,\boldsymbol{x},r).\end{aligned}$$

Hence from Darboux's equation (2.4.20), we reach the *Euler-Poisson-Darboux equation*:

$$\frac{1}{c^2}\partial_t^2 M_u(t,\boldsymbol{x},r) = \partial_r^2 M_u(t,\boldsymbol{x},r) + \frac{n-1}{r}\partial_r M_u(t,\boldsymbol{x},r). \tag{2.4.22}$$

Unlike the Darboux's equation (2.4.20), now the Euler-Poisson-Darboux equation (2.4.22) does not involve any derivaties on $\boldsymbol{x}$. Therefore we can refer $\boldsymbol{x} \in \mathbb{R}^n$ simply as a parameter here, and (2.4.22) is just a simple PDE involving only two variables (one time variable t and one spatial variable r).

In the particular case when $n = 3$, the computations are extremely easy: We multiply $c^2 r$ on (2.4.22) to obtain

$$\partial_t^2\left(rM_u(t,\boldsymbol{x},r)\right) = c^2\left(r\partial_r^2 M_u(t,\boldsymbol{x},r) + 2\partial_r M_u(t,\boldsymbol{x},r)\right) = c^2\partial_r^2\left(rM_u(t,\boldsymbol{x},r)\right),$$

this means that, for each parameter $\boldsymbol{x} \in \mathbb{R}^3$, $(t,r) \mapsto rM_u(t,\boldsymbol{x},r)$ satisfies the 1-dimensional wave equation. From (2.4.17), the initial conditions can be verified as well:

$$\begin{aligned} rM_u(0,\boldsymbol{x},r) &= rM_{u(0,\cdot)}(\boldsymbol{x},r) = rM_\phi(\boldsymbol{x},r), \\ \partial_t\left(rM_u(t,\boldsymbol{x},r)\right)\big|_{t=0} &= rM_{\partial_t u(0,\cdot)}(\boldsymbol{x},r) = rM_\psi(\boldsymbol{x},r),\end{aligned}$$

where the dominated convergence theorem (Lemma 1.0.2) involved in second identity. Therefore the d'Alembert formula (2.4.6) gives

$$\begin{aligned} rM_u(t,\boldsymbol{x},r) = \frac{1}{2}&\left((r+ct)M_\phi(t,\boldsymbol{x},r+ct) + (r-ct)M_\phi(t,\boldsymbol{x},r-ct)\right) \\ &+ \frac{1}{2c}\int_{r-ct}^{r+ct} sM_\psi(\boldsymbol{x},s)\,\mathrm{d}s.\end{aligned}$$

Since $M_\phi(\boldsymbol{x},\cdot)$ and $M_\psi(\boldsymbol{x},\cdot)$ are both even for each fixed $\boldsymbol{x}\in\mathbb{R}^3$, then one sees that

$$\begin{aligned}
&M_u(t,\boldsymbol{x},r)\\
&\quad= \frac{1}{2cr}\int_{ct-r}^{ct+r} sM_\psi(\boldsymbol{x},s)\,\mathrm{d}s + \frac{1}{2r}\left((ct+r)M_\phi(t,\boldsymbol{x},ct+r)-(ct-r)M_\phi(t,\boldsymbol{x},ct-r)\right)\\
&\quad= \frac{1}{c}\fint_{ct-r}^{ct+r} sM_\psi(\boldsymbol{x},s)\,\mathrm{d}s + \frac{(\tau M_\phi(t,\boldsymbol{x},\tau))|_{\tau=ct+r}-(\tau M_\phi(t,\boldsymbol{x},\tau))|_{\tau=ct-r}}{2r}.
\end{aligned}$$

Note that the first term is simply the difference quotient for differentiation, while the second term is the mean value term. Therefore, by letting $r\to 0$, we reach

$$\begin{aligned}
u(t,\boldsymbol{x}) &= \frac{1}{c}\,\tau M_\psi(t,\boldsymbol{x},\tau)|_{\tau=ct} + \partial_\tau\left(\tau M_\phi(t,\boldsymbol{x},\tau)\right)|_{\tau=ct}\\
&= tM_\psi(t,\boldsymbol{x},ct) + \partial_t\left(tM_\phi(t,\boldsymbol{x},ct)\right)\\
&= \frac{t}{\mathscr{H}^2(\partial B_{ct}(\boldsymbol{x}))}\int_{\partial B_{ct}(\boldsymbol{x})}\psi(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}} + \partial_t\left(t\frac{t}{\mathscr{H}^2(\partial B_{ct}(\boldsymbol{x}))}\int_{\partial B_{ct}(\boldsymbol{x})}\phi(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}}\right)\\
&= \frac{1}{4\pi c^2 t}\int_{|\boldsymbol{y}-\boldsymbol{x}|=ct}\psi(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}} + \partial_t\left(\frac{1}{4\pi c^2 t}\int_{|\boldsymbol{y}-\boldsymbol{x}|=ct}\phi(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}}\right) \qquad \text{(2.4.23a)}
\end{aligned}$$

because $\mathscr{H}^2(\partial B_{ct}(\boldsymbol{x})) = 4\pi(ct)^2$. By using (2.4.19), we also can write (2.4.23a) as

$$\begin{aligned}
u(t,\boldsymbol{x}) &= tM_\psi(t,\boldsymbol{x},ct) + \partial_t\left(tM_\phi(t,\boldsymbol{x},ct)\right)\\
&= \frac{t}{\omega_n}\int_{\mathcal{S}^{n-1}}\psi(\boldsymbol{x}+ct\hat{\boldsymbol{z}})\,\mathrm{d}\hat{\boldsymbol{z}} + \partial_t\left(\frac{t}{\omega_n}\int_{\mathcal{S}^{n-1}}\phi(\boldsymbol{x}+ct\hat{\boldsymbol{z}})\,\mathrm{d}\hat{\boldsymbol{z}}\right). \qquad \text{(2.4.23b)}
\end{aligned}$$

The solution (2.4.23a)., or equivalently (2.4.23b), is called the *Kirchhoff's formula*. We now reach the following lemma.

Lemma (Uniqueness). *Let $n=3$, and let $u\in C^2(\mathbb{R}^3)$ be a solution of the initial-value problem (2.4.17). Then u must in the Kirchhoff's formula (2.4.23a), or equivalently (2.4.23b).*

On the other hand, by assuming higher regularity initial conditions $\psi\in C^2(\mathbb{R}^3)$ and $\phi\in C^3(\mathbb{R}^3)$, the dominated convergence theorem (Lemma 1.0.2) guarantees that $u\in C^2(\mathbb{R}^3)$ and it satisfies (2.4.17) with $n=3$. Therefore we conclude the following theorem:

Theorem 2.4.13. *Let $n=3$, and let $\psi\in C^2(\mathbb{R}^3)$ and $\phi\in C^3(\mathbb{R}^3)$. Then the function u given by the Kirchhoff's formula (2.4.23a), or equivalently (2.4.23b), is the unique C^2-solution of the initial-value problem (2.4.17).*

Remark. The above theorem do not guarantee the initial data of C^2-solution u of the 3-dimensional wave equation $\partial_t^2 u(t,\boldsymbol{x})-\Delta u(t,\boldsymbol{x})=0$ in $\mathbb{R}^3$ must satisfies $u(0,\cdot)\in C^3(\mathbb{R}^3)$ and $\partial_t u(0,\cdot)\in C^2(\mathbb{R}^3)$.

Again, (2.4.23a), or equivalently (2.4.23b), defines a weak solution of the initial-value problem, since they are well-defined for $\psi\in C^0(\mathbb{R}^3)$ and $\phi\in C^1(\mathbb{R}^3)$.

Exercise 2.4.14 (Inhomogeneous 3D wave equation). Let $f\in C^2((0,\infty)\times\mathbb{R}^3)$, $\psi\in C^2(\mathbb{R}^3)$ and $\phi\in C^3(\mathbb{R}^3)$. Find the unique C^2-solution of the following initial-value problem:

$$\begin{cases}\partial_t^2 u(t,\boldsymbol{x})-\Delta u(t,\boldsymbol{x}) = f(t,\boldsymbol{x}) & \text{for } (t,\boldsymbol{x})\in(0,\infty)\times\mathbb{R}^3,\\ u(0,\boldsymbol{x})=\phi(\boldsymbol{x}),\quad \partial_t u(0,\boldsymbol{x})=\psi(\boldsymbol{x}) & \text{for } \boldsymbol{x}\in\mathbb{R}^3,\end{cases}$$

using Duhamel principle (Section 2.4.4).

We now consider the case when $n = 2$. In this case, rather directly compute the solution from the Euler-Poisson-Darboux equation (2.4.22), it is more easy to compute the solution from the the Kirchhoff's formula (2.4.23a), or equivalently (2.4.23b). This method is called the *Hadamard's method of descent*. Suppose that u is a solution of (2.4.17) with $n = 2$:

$$\begin{cases} \partial_t^2 u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = 0 & \text{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^2, \\ u(0,\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x}\in\mathbb{R}^2, \end{cases}$$

with $\phi \in C^3(\mathbb{R}^2)$ and $\psi \in C^2(\mathbb{R}^2)$. We now introduce an artificial parameter x_3, and write $\tilde{\boldsymbol{x}} = (\boldsymbol{x}, x_3) = (x_1, x_2, x_3)$. Accordingly, we consider the extended operator $\tilde{\Delta} = \Delta + \partial_3^2 = \partial_1^2 + \partial_2^2 + \partial_3^2$, and such u satisfies

$$\begin{cases} \partial_t^2 u(t,\tilde{\boldsymbol{x}}) - \Delta u(t,\tilde{\boldsymbol{x}}) = 0 & \text{for } (t,\tilde{\boldsymbol{x}}) \in (0,\infty)\times\mathbb{R}^3, \\ u(0,\tilde{\boldsymbol{x}}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\tilde{\boldsymbol{x}}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x}\in\mathbb{R}^3, \end{cases}$$

therefore the unique C^2-solution is given by the Kirchhoff's formula (2.4.23a):

$$\begin{aligned} u(t,\boldsymbol{x}) &= \frac{1}{4\pi c^2 t}\int_{|\tilde{\boldsymbol{y}}-(\boldsymbol{x},0)|=ct} \psi(\tilde{\boldsymbol{y}})\,\mathrm{d}S_{\tilde{\boldsymbol{y}}} + \partial_t\left(\frac{1}{4\pi c^2 t}\int_{|\tilde{\boldsymbol{y}}-(\boldsymbol{x},0)|=ct} \phi(\tilde{\boldsymbol{y}})\,\mathrm{d}S_{\tilde{\boldsymbol{y}}}\right) \\ &= \frac{1}{4\pi c^2 t}\int_{|\tilde{\boldsymbol{y}}-(\boldsymbol{x},0)|=ct} \psi(\boldsymbol{y})\,\mathrm{d}S_{\tilde{\boldsymbol{y}}} + \partial_t\left(\frac{1}{4\pi c^2 t}\int_{|\tilde{\boldsymbol{y}}-(\boldsymbol{x},0)|=ct} \phi(\boldsymbol{y})\,\mathrm{d}S_{\tilde{\boldsymbol{y}}}\right). \end{aligned} \tag{2.4.24}$$

In order to simplify the above expression, we need the following well-known fact:

LEMMA 2.4.15 (Surface integral). *Let S be a surface in $\mathbb{R}^3$, which can be parameterized as*

$$\boldsymbol{r}(s_1,s_2) = (r_1(s_1,s_2), r_2(s_1,s_2), r_3(s_1,s_2))$$

where $r_1, r_2, r_3 \in C^1(\overline{\Omega})$ for some Jordan-measurable open set $\Omega \subset \mathbb{R}^2$. Then one has

$$\int_S \phi(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} = \int_\Omega \phi(\boldsymbol{r}(s_1,s_2))V(s_1,s_2)\,\mathrm{d}(s_1,s_2),$$

where the volume form V is given by

$$V(s_1,s_2) = \sqrt{g_{11}g_{22} - g_{12}^2} = |\partial_{s_1}\boldsymbol{r}(s_1,s_2)\times\partial_{s_2}\boldsymbol{r}(s_1,s_2)|$$

where $g_{ij} = \partial_{s_i}\boldsymbol{r}(s_1,s_2)\cdot\partial_{s_j}\boldsymbol{r}(s_1,s_2)$ for $i,j\in\{1,2\}$.

REMARK 2.4.16. In fact, g_{ij} are the coefficients of the *first fundamental form*. If we choose $r_1(s_1,s_2) = s_1$, $r_2(s_1,s_2) = s_2$ and $r_3(s_1,s_2) = f(s_1,s_2)$, we compute that

$$\partial_{s_1}\boldsymbol{r}(s_1,s_2) = (1,0,\partial_{s_1}f(s_1,s_2)), \quad \partial_{s_2}\boldsymbol{r}(s_1,s_2) = (0,1,\partial_{s_2}f(s_1,s_2)),$$

and thus

$$g_{11} = 1 + |\partial_{s_1}f(s_1,s_2)|^2, \quad g_{22} = 1 + |\partial_{s_2}f(s_1,s_2)|^2, \quad g_{12} = \partial_{s_1}f(s_1,s_2)\partial_{s_2}f(s_1,s_2).$$

Therefore the volume form is given by

$$V(s_1,s_2) = \sqrt{1 + |\partial_{s_1}f(s_1,s_2)|^2 + |\partial_{s_2}f(s_1,s_2)|^2}.$$

We now continue simplify (2.4.24). We now split the sphere $\{\tilde{\boldsymbol{y}}\in\mathbb{R}^3 : |\tilde{\boldsymbol{y}} - (\boldsymbol{x},0)| = ct\}$ into two halfs, each can be parametrized as

$$f_\pm(y_1,y_2) = \pm\sqrt{c^2t^2 - |\boldsymbol{y}-\boldsymbol{x}|^2} = \pm\sqrt{c^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}.$$

We compute

$$\partial_1 f_\pm(y_1,y_2) = \mp\frac{y_1-x_1}{\sqrt{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}}, \quad \partial_2 f_\pm(x_1,x_2) = \mp\frac{y_2-x_2}{\sqrt{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}},$$

thus the volume form is given by

$$\begin{aligned} V_\pm(y_1,y_2) &= \sqrt{1+\frac{(y_1-x_1)^2}{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}+\frac{(y_2-x_2)^2}{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}} \\ &= \sqrt{\frac{c^2t^2}{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}} = \frac{ct}{\sqrt{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}}, \end{aligned}$$

therefore (2.4.24) becomes
(2.4.25)

$$u(t,\boldsymbol{x}) = \frac{1}{2\pi c}\int_{|\boldsymbol{y}-\boldsymbol{x}|\le ct}\frac{\psi(\boldsymbol{y})}{\sqrt{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}}\,\mathrm{d}\boldsymbol{y} + \partial_t\left(\frac{1}{2\pi c}\int_{|\boldsymbol{y}-\boldsymbol{x}|\le ct}\frac{\phi(\boldsymbol{y})}{\sqrt{c^2t^2-|\boldsymbol{y}-\boldsymbol{x}|^2}}\,\mathrm{d}\boldsymbol{y}\right).$$

It is interesting to compact the 3D-wave (2.4.23a) and 2D-wave equation (2.4.25).

(1) **3D wave.** In order to compute the value $u(t,x_1,x_2,x_3)$, we used the values

$$\left\{(\psi(y_1,y_2,y_3),\phi(y_1,y_2,y_3)) : \sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2} = ct\right\}.$$

In other words, the *domain of dependence* of 3D wave is *on* the cone

$$\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2} = ct,$$

this is called the *Huygen's principle.*

(2) **2D wave.** In order to compute the value $u(t,x_1,x_2)$, we used the values

$$\left\{(\psi(y_1,y_2),\phi(y_1,y_2)) : \sqrt{(y_1-x_1)^2+(y_2-x_2)^2} \le ct\right\}.$$

In other words, the domain of dependence of 2D wave is *in* the cone

$$\sqrt{(y_1-x_1)^2+(y_2-x_2)^2} \le ct,$$

and thus the Huygen's principle fails.

EXERCISE 2.4.17 (Inhomogeneous 2D wave equation). Let $f \in C^2((0,\infty)\times\mathbb{R}^2)$, $\psi \in C^2(\mathbb{R}^2)$ and $\phi \in C^3(\mathbb{R}^2)$. Find the unique C^2-solution of the following initial-value problem:

$$\begin{cases} \partial_t^2 u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = f(t,\boldsymbol{x}) & \text{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^2, \\ u(0,\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x}\in\mathbb{R}^2, \end{cases}$$

using Duhamel principle (Section 2.4.4).

EXERCISE 2.4.18 (5D wave equation, [**Joh78**, pages 109–110]). Consider the initial-value problem (2.4.17) with $n=5$. Let $M_u(t,\boldsymbol{x},r)$ given in (2.4.19), which satisfies the Darboux equation (2.4.20), set

$$N_u(t,\boldsymbol{x},r) = r^2\partial_r M_u(t,\boldsymbol{x},r) + 3rM_u(t,\boldsymbol{x},r).$$

(a) Show that $N_u(t,\boldsymbol{x},r)$ is a solution of $\partial_t^2 N_u = c^2\partial_r^2 N_u$ and find N_u from its initial data in terms of $M_f(\boldsymbol{x},r)$ and $M_g(\boldsymbol{x},r)$.

(b) Show that

$$u(t,\boldsymbol{x}) = \lim_{r\to 0} \frac{N_u(t,\boldsymbol{x},r)}{3r} = \left(\frac{1}{3}t^2\partial_t + t\right) M_g(ct,\boldsymbol{x}) + \partial_t \left(\frac{1}{3}t^2\partial_t + t\right) M_f(ct,\boldsymbol{x}).$$

EXERCISE 2.4.19 (4D wave equation, [**Joh78**, page 112]). Solve the initial-value problem (2.4.17) with $n = 4$. [Hint: Hadamard's method of descent]

REMARK 2.4.20. In fact, the Huygen's principle holds true for all odd dimension wave equations, but fails for all even dimension wave equations.

CHAPTER 3

Partial differential equation in weak sense

3.1. Weak derivatives and distribution derivatives

In practical application, we should expect there are singularities in solution, for example:

(1) the general solution of transport equation (2.2.3), which in general need not to be C^1;
(2) the general solution of 1D wave equation, i.e. d'Alembert formula (2.4.6), which in general need not to be C^2;
(3) the general solution of 3D wave equation, i.e. Kirchhoff's formula (2.4.23a), which in general need not to be C^2.

One "simplest" way to interpret "weak solutions" is directly write down the explicit solution. However, this idea is difficult in general. Therefore we need some systematic way to interpret the "weak solutions".

We first recall some materials from my previous lecture notes on Fourier analysis course [**Kow22**]. Let Ω be any open set in $\mathbb{R}^n$. By using integration by parts formula (Corollary 1.0.19), one has

$$\int_\Omega (\partial_j f)\varphi \,\mathrm{d}\boldsymbol{x} = -\int_\Omega f\partial_j\varphi \,\mathrm{d}\boldsymbol{x} \quad \text{for all } f,\varphi \in C_c^1(\Omega),$$

where $C_c^k(\Omega)$ is given in (1.0.5). In fact, we have:

Exercise 3.1.1. Show that

$$\int_\Omega (\partial^{\boldsymbol{\alpha}} f)\varphi \,\mathrm{d}\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|}\int_\Omega f\partial^{\boldsymbol{\alpha}}\varphi \,\mathrm{d}\boldsymbol{x} \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

for all multi-indices α.

Therefore, it is quite natural to consider the following definition (and we can interpret the PDE using the following weak derivatives):

Definition 3.1.2 (Weak derivatives). We define the locally-L^1 space by

$$L^1_{\rm loc}(\Omega) := \left\{ f \text{ defined on } \Omega : \|f\|_{L^1(K)} := \int_K |f(\boldsymbol{x})|\mathrm{d}\boldsymbol{x} < \infty \text{ for all compact set } K\subset\Omega \right\}.$$

A function $g \in L^1_{\rm loc}(\Omega)$ (if exists) is called a *weak derivative* of $f \in L^1_{\rm loc}(\Omega)$ (of order $\boldsymbol{\alpha}$) if

$$\int_\Omega g\varphi \,\mathrm{d}\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|}\int_\Omega f\partial^{\boldsymbol{\alpha}}\varphi \,\mathrm{d}\boldsymbol{x} \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

and we often denote g as $\partial^{\boldsymbol{\alpha}} f$.

The well-definedness of the weak derivatives (i.e. each function $g \in L^1_{\rm loc}(\Omega)$ produced from $f \in L^1_{\rm loc}(\Omega)$ must be unique) is guaranteed by the following lemma:

LEMMA 3.1.3 (Uniqueness of weak derivatives [**Mit18**, Theorem 1.3]). *If $g \in L^1_{\rm loc}(\Omega)$ satisfying $g = 0$ in Ω in distribution sense, i.e.*

$$\int_\Omega g\varphi \, \mathrm{d}\boldsymbol{x} = 0 \quad \textit{for all } \varphi \in C_c^\infty(\Omega),$$

then $g = 0$ a.e. in Ω.

REMARK 3.1.4. The converse of Lemma 3.1.3 is trivial. Here and after, we shall omit the notation "a.e." if there is no any ambiguity. The above lemma only guarantee uniqueness, but not existence.

We now give a first example of weak derivatives.

EXAMPLE 3.1.5. We consider the Heaviside function

$$H(x) := \begin{cases} 1 & \text{for all } x > 0, \\ 0 & \text{for all } x \le 0. \end{cases} \tag{3.1.1}$$

It is easy to see that $H \in L^1_{\rm loc}(\mathbb{R})$. We define

$$f(x) := \begin{cases} x & \text{for all } x > 0, \\ 0 & \text{for all } x \le 0. \end{cases}$$

One can verify that

$$-\int_{\mathbb{R}} f(x)\varphi'(x) \, \mathrm{d}x = -\int_0^\infty x\varphi'(x) \, \mathrm{d}x = -x\varphi(x)|_{x=0}^{x\to\infty} + \int_0^\infty \varphi(x) \, \mathrm{d}x = \int_{\mathbb{R}} H(x)\varphi(x) \, \mathrm{d}x,$$

which shows that the Heaviside function (3.1.1) is the weak derivative of f (of order 1), and we simply denote $f' = H$.

However, not all $L^1_{\rm loc}(\Omega)$ function admits weak derivatives:

EXAMPLE 3.1.6. We now show that the weak derivative of the Heaviside function H given in (3.1.1) of order 1 does not exist. Suppose the contrary, that H has a weak derivative of order 1, says $g \in L^1_{\rm loc}(\mathbb{R})$. By Definition 3.1.2, we see that

$$\int_{-\infty}^\infty g(x)\varphi(x) \, \mathrm{d}x = -\int_{-\infty}^\infty H(x)\varphi'(x) \, \mathrm{d}x = -\int_0^\infty \varphi'(x) \, \mathrm{d}x = \varphi(0) \tag{3.1.2}$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. Hence we know that

$$\int_{-\infty}^\infty g(x)\varphi(x) \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R} \setminus \{0\}).$$

By using Lemma 3.1.3 with $\mathbb{R} \setminus \{0\}$, we conclude that $g = 0$ a.e. in $\mathbb{R}$, and from (3.1.2) we have $\varphi(0) = 0$ for all $\varphi \in C_c^\infty(\mathbb{R})$, which is an obvious contradiction.

Since the weak derivatives may not exist, it is much more convenient to consider a generalization the weak derivatives, called the *distribution derivatives*. In order to do so, we need to explain what is a distribution (or *generalized functions*). Here we just give an introductory briefing, see e.g. my lecture note [**Kow22**] for more details. Fixing any compact set K in $\mathbb{R}^n$, we denote

$$\mathscr{D}_K := \{\varphi \in C^\infty(\mathbb{R}^n) : \mathrm{supp}\,(\varphi) \subset K\}.$$

DEFINITION 3.1.7. Given a vector space X over a subfield F of the complex numbers $\mathbb{C}$, a *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties:

(1) **Positive definiteness.** $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$;
(2) **Absolute homogeneity.** $\|sx\| = |s|\|x\|$ for all $x \in X$ and scalars $s \in F$; and
(3) **Subadditivity/Triangle inequality.** $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

In this case, we call the pair $(X, \|\cdot\|)$ the *normed space.* If a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ satisfies the following: Given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$ for all $n, m \geq n_0$, then we call such sequence $\{x_n\}$ a *Cauchy sequence.* If each Cauchy sequence converges in X (i.e. for each Cauchy sequence $\{x_n\} \subset X$ there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$), then we say that the normed space $(X, \|\cdot\|)$ is *complete* or *Banach.*

For each fixed $N \in \mathbb{Z}_{\geq 0}$, it is easy to see that

$$\|\varphi\|_{N,K} := \sum_{|\alpha|\leq N} \|\partial^\alpha \varphi\|_{L^\infty(K)}$$

is a norm defined on $\mathscr{D}_K$. However, the normed space $(\mathscr{D}_K, \|\cdot\|_{N,K})$ is not complete, since the norm does not involve derivatives of order $> N$. We can further generalize the notion "norm" in the following definition:

DEFINITION 3.1.8. Given a set M, a *metric* is a function $\mathsf{d} : M \times M \to \mathbb{R}$ with the following properties:

(1) **Positive definiteness.** $\mathsf{d}(x, y) \geq 0$ for all $x, y \in M$ and $\mathsf{d}(x, y) = 0$ iff $x = y$;
(2) **Symmetry.** $\mathsf{d}(x, y) = \mathsf{d}(y, x)$ for all $x, y \in M$; and
(3) **Subadditivity/Triangle inequality.** $\mathsf{d}(x, z) \leq \mathsf{d}(x, y) + \mathsf{d}(y, z)$ for all $x, y, z \in M$.

In this case, we call the pair (M, d) the *metric space.* If a sequence $\{x_n\}_{n\in\mathbb{N}} \subset M$ satisfies the following: Given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathsf{d}(x_n, x_m) < \epsilon$ for all $n, m \in n_0$, then we call such sequence $\{x_n\}$ a *Cauchy sequence.* If each Cauchy sequence converges in M (i.e. for each Cauchy sequence $\{x_n\} \subset M$ there exists $x \in M$ such that $\mathsf{d}(x_n, x) \to 0$ as $n \to \infty$), then we say that the metric space (M, d) is *complete* or *Fréchet.*

EXERCISE 3.1.9. Show that each normed space $(X, \|\cdot\|)$ is also a metric space.

LEMMA 3.1.10. *$\mathscr{D}_K$ is Fréchet equipped with the metric*

$$\mathsf{d}_{\mathscr{D}_K}(\varphi, \psi) := \sum_{N=0}^{\infty} 2^{-N} \frac{\|\varphi - \psi\|_{N,K}}{1 + \|\varphi - \psi\|_{N,K}}. \tag{3.1.3}$$

REMARK 3.1.11. Since $\mathscr{D}_K$ is a (Grothedieck) nuclear space, then it is not possible to find a norm which is complete. In other words, if we equipped $\mathscr{D}_K$ by any norm $\|\cdot\|$, then $(\mathscr{D}_K, \|\cdot\|)$ cannot be Banach.

EXERCISE 3.1.12. Verify that $\mathsf{d}_{\mathscr{D}_K}(\varphi, \psi)$ is a metric.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We now define the set of *test functions* by

$$\mathscr{D}(\Omega) := \bigcup_{K\subset\Omega \text{ compact}} \mathscr{D}_K.$$

If we view $\mathscr{D}(\Omega)$ as a set, then it is easy to see that $\mathscr{D}(\Omega) = C_c^\infty(\Omega)$ given in (1.0.5). Of course, one can equipped $\mathscr{D}(\Omega)$ by the metric

$$\mathsf{d}_{\mathscr{D}(\Omega)}(\varphi,\psi) := \sum_{N=0}^{\infty} 2^{-N} \frac{\|\varphi-\psi\|_{N,\Omega}}{1+\|\varphi-\psi\|_{N,\Omega}}, \quad \|\varphi\|_{N,\Omega} := \sum_{|\alpha|\le N} \|\partial^\alpha \varphi\|_{L^\infty(\Omega)}.$$

However, unlike Lemma 3.1.10, $(\mathscr{D}(\Omega), \mathsf{d}_{\mathscr{D}(\Omega)})$ is not complete, see the following exercise:

EXERCISE 3.1.13. Take $n = 1$ and $\Omega = \mathbb{R}$. Let $\phi \in \mathscr{D}(\mathbb{R})$ with $\operatorname{supp}(\phi) \subset [0,1]$ and $\phi > 0$ in $(0,1)$. Define

$$\psi_m(x) := \phi(x-1) + \frac{1}{2}\phi(x-2) + \cdots + \frac{1}{m}\phi(x-m).$$

Show that $\{\psi_m\}$ is a Cauchy sequence in $(\mathscr{D}(\mathbb{R}), \mathsf{d}_{\mathscr{D}(\mathbb{R})})$, but the limit $\lim_{m\to\infty} \psi_m(x)$ does not have compact support.

We now introduce the following general notion:

DEFINITION 3.1.14. Given a set X, a *topology* τ is a collection of subsets of X with the following properties:

(1) $\emptyset, X \in \tau$;
(2) Any union of elements of τ is also an element of τ; and
(3) Any finite intersection of elements of τ is also an element of τ.

If τ is a topology on X, then the pair (X, τ) is called a *topology space*. Each element in τ is called an *open set* of X (with respect to τ).

We usually equip $\mathscr{D}(\Omega)$ by another (locally convex) topology τ in which "Cauchy sequence" do converge, i.e. "complete". Here we will not go through these definitions as well as technical details, here we refer to [**Rud91**, Chapter 6] or [**Mit18**, Appendix 14.1] for details.

DEFINITION 3.1.15. We refer the linear mapping $T : (\mathscr{D}(\Omega), \tau) \to (\mathbb{R}, |\cdot|)$ as the *linear functional* on $(\mathscr{D}(\Omega), \tau)$. Let D be an open set in $\mathbb{R}$, we define the preimage by

$$T^{-1}(D) := \{ f \in \mathscr{D}(\Omega) : T(f) \in D \}.$$

We called such linear functional T is *continuous* (with respect to τ) if $T^{-1}(D) \in \tau$ for each open set D. The set of continuous linear functionals on $(\mathscr{D}(\Omega), \tau)$ is denoted by $\mathscr{D}'(\Omega)$ and its elements are called *distributions* on Ω.

The following lemma, which is a special case of [**Rud91**, Chapter 6] or [**Mit18**, Appendix 14.6], gives equivalent characterization of continuity of linear functionals on $\mathscr{D}(\Omega)$:

LEMMA 3.1.16. *Let T be a linear functional on $(\mathscr{D}(\Omega), \tau)$, then the following are equivalent:*

(1) *T is continuous with respect to τ;*
(2) *$\lim_{j\to\infty} T(\varphi_j) = 0$ whenever $\varphi_j \to 0$ in $(\mathscr{D}(\Omega), \tau)$;*
(3) *$T|_{\mathscr{D}_K}$ is continuous for each compact set $K \subset \Omega$ with respect to the metric d_K given in (3.1.3).*

For simplicity, we usually denote $\mathscr{D}(\Omega)$, or even $C_c^\infty(\Omega)$, to represent the topological space $(\mathscr{D}(\Omega), \tau)$, as we will not focus on its topological aspect in this lecture note.

EXAMPLE 3.1.17. Each element $f \in L^1_{\rm loc}(\Omega)$ can be identify with $T_f \in \mathscr{D}'(\Omega)$ defined by

$$T_f(\varphi) := \int_{\mathbb{R}^n} f(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

with the estimate

$$|T_f(\varphi)| \leq \int_K |f(\boldsymbol{x})\varphi(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \leq \|\varphi\|_{L^\infty(K)} \int_K |f(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \quad \text{for all } \varphi \in \mathscr{D}_K.$$

for all compact subset $K \subset \Omega$. Therefore one can simply write $L^1_{\rm loc}(\Omega) \subset \mathscr{D}'(\Omega)$.

EXAMPLE 3.1.18 (Dirac measure). Fix a point $\boldsymbol{x}_0 \in \Omega$ and one can verify that the linear functional

$$T(\varphi) := \varphi(\boldsymbol{x}_0)$$

is indeed continuous, i.e. $T \in \mathscr{D}'(\Omega)$. We usually denote such distribution T by $\delta_{\boldsymbol{x}_0}$. However, $\delta_{\boldsymbol{x}_0} \notin L^1_{\rm loc}(\Omega)$ if we consider the identification given in Example (3.1.17): Suppose the contrary, there exists $f \in L^1_{\rm loc}(\Omega)$ such that

$$\varphi(\boldsymbol{x}_0) = \delta_{\boldsymbol{x}_0}(\varphi) = \int_\Omega f(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

If we choose $\varphi \in C_c^\infty(\Omega \setminus \{\boldsymbol{x}_0\})$, then we conclude that $f = 0$ a.e. in Ω, which is immediately a contradiction.

In view of Definition 3.1.2 and Example 3.1.17, we now able to define the distirbution derivatives:

DEFINITION 3.1.19. For any $T \in \mathscr{D}'(\Omega)$, the *distribution derivative* $\partial^\alpha T \in \mathscr{D}'(\Omega)$ of T is defined by

$$(\partial^\alpha T)(\varphi) := (-1)^{|\alpha|} T(\partial^\alpha \varphi) \quad \text{for all } \varphi \in \mathscr{D}(\Omega).$$

Unlike weak derivatives (Definition 3.1.2), *distribution derivative always exist.*

EXAMPLE 3.1.20. Let H be the Heaviside function given in (3.1.1). Since $H \in L^1_{\rm loc}(\mathbb{R})$, then we can identify it with the distribution T_H as in Example 3.1.17. By definition, one sees that its distributional derivative of order 1, here we denoted by T_H', is given by

$$T_H'(\varphi) = -T_H(\varphi') = -\int_{\mathbb{R}^n} H(x)\varphi'(x)\,\mathrm{d}x = -\int_0^\infty \varphi'(x)\,\mathrm{d}x = \varphi(0) = \delta_0(\varphi)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, which means that δ_0 is its distributional derivative. It is interesting to compare this with Example 3.1.6, which shows that the weak derivative of H does not exist.

From now on, we will denote ∂^α the distribution derivative without explicitly mentioning. In addition, we will use the term "derivative", "differentiation" without mention "distribution sense" explicitly.

EXERCISE 3.1.21. Prove that for every $c \in \mathbb{R}$ one has

$$(e^{-c|x|})' = -ce^{-cx}H(x) + ce^{cx}H(-x) \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

EXERCISE 3.1.22. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \ln |x| - x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that f is a continuous function and compute its distributional derivative f'.

3.2. Definition and elementary properties of the Sobolev spaces

We first recall the following definition, which appeared in Theorem 3.2.8 above:

Definition 3.2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$. For each $m \in \mathbb{N}$, we define the Sobolev spaces

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \leq m\}.$$

In fact, $W^{m,p}(\Omega)$ is a normed space with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ given by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$
$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^\infty(\Omega)}.$$

The elements in $W^{m,p}(\Omega)$ also called the *Sobolev functions*.

Lemma 3.2.2 ([**AH09**, Theorem 7.2.3]). *Let $1 \leq p \leq \infty$, let $m \in \mathbb{N}$ and let Ω be an open set in $\mathbb{R}^n$. Then the Sobolev space $W^{m,p}(\Omega)$ is Banach.*

Similar to L^p-functions, Sobolev functions also can be approximated by smooth functions.

Lemma 3.2.3 ([**AH09**, Theorem 7.3.1]). *Let Ω be an open set in $\mathbb{R}^n$, let $1 \leq p < \infty$ and let $m \in \mathbb{N}$. Given any $f \in W^{m,p}(\Omega)$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ such that*

$$f_n \to f \text{ in } W^{m,p}(\Omega), \quad \text{that is,} \quad \lim_{k\to\infty} \|f_k - f\|_{W^{m,p}(\Omega)} = 0.$$

Note that in this lemma, the approximation functions $\{f_k\}$ are smooth *only in the interior* of Ω. To have the smoothness *up to the boundary* of the approximation sequence, we need to make a smoothness assumption on the boundary $\partial\Omega$. For each $k \in \mathbb{N} \cup \{\infty\}$, we define

$$C^k(\overline{\Omega}) := \left\{f|_{\overline{\Omega}} : f \in C^k(U) \text{ for some open set } U \supset \overline{\Omega}\right\}.$$

Lemma 3.2.4 ([**AH09**, Theorem 7.3.2]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, let $1 \leq p < \infty$ and let $m \in \mathbb{N}$. Given any $f \in W^{m,p}(\Omega)$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ such that*

$$f_n \to f \text{ in } W^{m,p}(\Omega), \quad \text{that is,} \quad \lim_{k\to\infty} \|f_k - f\|_{W^{m,p}(\Omega)} = 0.$$

Before introducing the Sobolev embeddings, we first introducing the following concept:

Definition 3.2.5. Let X and Y be two Banach spaces. We say that the space X is *continuous embedded* in Y if

$$\|v\|_Y \leq c\|v\|_X \quad \text{for all } v \in X. \tag{3.2.1}$$

We say that the space X is *compactly embedded* in Y if (3.2.1) holds and each bounded sequence in X has a convergent subsequence in Y.

Many authors (including myself) simply denote $X \subset Y$ if the Banach space X is continuous embedded in another Banach space Y, despite that X is not necessarily a subset of Y. We will also denote $X \Subset Y$ if X is compactly embedded in Y. Here and after (including the next theorem), we will use these notations without mentioning explicitly. Let $\lfloor x \rfloor$ denotes the integer part of x, and we have the following theorem:

THEOREM 3.2.6 (Sobolev embedding theorems [**AH09**, Theorem 7.3.7 and Theorem 7.3.8]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then the following statements are valid:*

(a) *If $k < \frac{n}{p}$, then $W^{k,p}(\Omega) \Subset L^q(\Omega)$ for any $q < p^*$ and $W^{k,p}(\Omega) \subset L^q(\Omega)$ when $q \le p^*$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}$.*

(b) *If $k = \frac{n}{p}$, then $W^{k,p}(\Omega) \Subset L^q(\Omega)$ for any $q < \infty$.*

(c) *If $k > \frac{n}{p}$, then*

$$W^{k,p}(\Omega) \Subset C^{k-\lfloor \frac{n}{p} \rfloor -1,\beta}(\Omega) \quad \textit{for all } \beta \in \left[0, \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}\right)$$

$$W^{k,p}(\Omega) \subset C^{k-\lfloor \frac{n}{p} \rfloor -1,\beta}(\Omega) \quad \textit{with } \beta = \begin{cases} \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p} & \textit{if } \frac{n}{p} \notin \mathbb{Z}, \\ \textit{any positive number } < 1 & \textit{if } \frac{n}{p} \in \mathbb{Z}. \end{cases}$$

REMARK 3.2.7. Theorem 3.2.6 is also valid for $W^{k,p}$-spaces with $k \in \mathbb{R}$, see e.g. [**AH09, McL00**] for precise definitions. Here we will cover these topics in this lecture note. Part (c) of Theorem 3.2.6 in particular gives some sufficient condition in terms of weak derivatives to guarantee the well-definedness of the strong/classical derivatives.

In fact, the integration by parts also holds true for weak derivatives (which generalized Corollary 1.0.19):

THEOREM 3.2.8 (Integration by parts [**EG15**, Theorem 1 in Section 4.3]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and given $1 \le p < \infty$. The mapping*

$$\mathrm{Tr} : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega), \quad \mathrm{Tr}\,(f) = f|_{\partial\Omega} \tag{3.2.2}$$

can be uniquely extended to a bounded surjective linear operator $W^{1,p}(\Omega) \to \mathrm{Tr}\,(W^{1,p}(\Omega)) \subset L^p(\partial\Omega)$. Furthermore, for all $\boldsymbol{\varphi} \in (C^1(\mathbb{R}^n))^n$ and $f \in W^{1,p}(\Omega)$, we have

$$\int_\Omega f \,\mathrm{div}\,(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla f \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{\nu}\cdot\boldsymbol{\varphi})\mathrm{Tr}\,(f)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{3.2.3}$$

where $\boldsymbol{\nu}$ is the unit outer normal to $\partial\Omega$.

REMARK 3.2.9. Here we refer the advance monograph [**EG15**] for the precise meaning of $\boldsymbol{\nu}$, which is well-defined for $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$. The function $\mathrm{Tr}(f)$ given in (3.2.2) is called the *trace* of f on $\partial\Omega$. We usually still denote $\mathrm{d}\mathscr{H}^{n-1}$ by $\mathrm{d}S_{\boldsymbol{x}}$. If there is no ambituity, we sometime omit the notation the trace operator (3.2.2) and simply write (3.2.3) as

$$\int_\Omega f \,\mathrm{div}\,(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla f \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{\nu}\cdot\boldsymbol{\varphi}) f\,\mathrm{d}S_{\boldsymbol{x}}.$$

We finally end this section by giving some remarks on convolution. The following lemma exhibit the (strong) differentiability of convolution:

LEMMA 3.2.10 ([**Bre11**, Proposition 4.20]). *Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and let $f \in C_c^m(\mathbb{R}^n)$ for some integer $m \in \mathbb{Z}_{\ge 0}$. Then $f * g \in C^m(\mathbb{R}^n)$ and*

$$\partial^\alpha(f * g) = (\partial^\alpha f) * g \quad \textit{for all multiindices } \alpha \textit{ with } |\alpha| \le m.$$

The following lemma exhibits the (weak) differentiability of convolution:

LEMMA 3.2.11 ([**Bre11**, Lemma 9.1]). *Let $\rho \in L^1(\mathbb{R}^n)$ and let $v \in W^{m,p}(\mathbb{R}^n)$ with $1 \le p \le \infty$ and $m \in \mathbb{N}$. Then*

$$\rho * v \in W^{m,p}(\mathbb{R}^n) \quad \textit{and} \quad \partial^\alpha(\rho * v) = \rho * \partial^\alpha v \textit{ for all } \alpha \textit{ with } |\alpha| \le m.$$

3.3. Hilbert spaces

We begin this section by introducing the following definition:

DEFINITION 3.3.1. Let H be a vector space. We say that a mapping $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is a *bilinear form* if

$$\begin{aligned}&(\cdot, u) \text{ is linear for each fixed } u \in H,\\&(v, \cdot) \text{ is linear for each fixed } v \in H.\end{aligned}$$

A *scalar product* or *inner product* is a bilinear form $(\cdot,\cdot) : H \times H \to \mathbb{R}$ such that

(1) **Positive definiteness.** $(u, u) \geq 0$ for all $u \in H$ and $(u, u) = 0$ iff $u = 0$;
(2) **Symmetry.** $(u, v) = (v, u)$ for all $u, v \in H$; and

In this case, we call the pair $(H, (\cdot,\cdot))$ an *inner product space*.

EXERCISE 3.3.2. Show the following Cauchy-Schwartz inequality:

$$|(u, v)| \leq (u, u)^{\frac{1}{2}}(v, v)^{\frac{1}{2}} \quad \text{for all } u, v \in H.$$

In addition, show that the function $\|\cdot\|$ defined by $\|u\| := (u, u)^{\frac{1}{2}}$ for all $u \in H$ is a norm, which satisfies the parallelogram law:

$$\left\|\frac{u+v}{2}\right\|^2 + \left\|\frac{u-v}{2}\right\|^2 = \frac{1}{2}(\|u\|^2 + \|v\|^2) \quad \text{for all } u, v \in H. \tag{3.3.1}$$

DEFINITION 3.3.3. Given an inner product space $(H, (\cdot,\cdot))$, and induce a norm $\|\cdot\|$ as above. If $(H, \|\cdot\|)$ is complete, then we called H a *Hilbert space*.

EXAMPLE 3.3.4. Let Ω be any open set in $\mathbb{R}^n$, and we define $L^2(\Omega) := \left\{f : \Omega \to \mathbb{R} \text{ with } \int_\Omega |f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} < \infty\right\}$. The mapping

$$\begin{aligned}&(\cdot,\cdot)_{L^2(\Omega)} : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}\\&(u, v)_{L^2(\Omega)} := \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } u, v \in L^2(\Omega)\end{aligned}$$

is a scalar product, and we denote $\|u\|_{L^2(\Omega)} = \sqrt{(u, u)_{L^2(\Omega)}}$ for all $u \in L^2(\Omega)$ the corresponding norm. In fact, $L^2(\Omega)$ is complete with respect to the norm $\|\cdot\|_{L^2(\Omega)}$, see e.g. [**WZ15**].

EXERCISE 3.3.5 ([**Bre11**, Exercise 5.1]). Let $(H, \|\cdot\|)$ be a normed space. Suppose that the norm $\|\cdot\|$ satisfies the parallelogram law (3.3.1). Define

$$(u, v) := \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2) \quad \text{for all } u, v \in H.$$

(1) Check that $(u, u) = \|u\|^2$, $(u, v) = (v, u)$, $(-u, v) = -(u, v)$ and $(u, 2v) = 2(u, v)$ for all $u, v \in H$.
(2) Prove that $(u + v, w) = (u, w) + (v, w)$ for all $u, v, w \in H$. [Hint: use the parallelogram law successively with (i) $u = \tilde{u}$, $v = \tilde{v}$; (ii) $u = \tilde{u} + \tilde{w}$, $v = \tilde{v} + \tilde{w}$; and (iii) $u = \tilde{u} + \tilde{v} + \tilde{w}$, $v = \tilde{w}$]
(3) Prove that $(\lambda u, v) = \lambda (u, v)$ for all $\lambda \in \mathbb{R}$ and for all $u, v \in H$. [Hint: Consider first the case $\lambda \in \mathbb{N}$, then $\lambda \in \mathbb{Q}$, and finally $\lambda \in \mathbb{R}$]
(4) Conclude that $(\cdot,\cdot)$ is a scalar product on H.

EXERCISE 3.3.6 (L^p is not a Hilbert space for $p \neq 2$ [**Bre11**, Exercise 5.2]). Show that $\|f\|_{L^p(\Omega)}$ satisfies the parallelogram law (3.3.1) if and only if $p = 2$. [Hint: Use functions with disjoint supports]

The above exercise suggests us to denote the following notations:

DEFINITION 3.3.7. Let Ω be any open set in $\mathbb{R}^n$, then we denote $H^m(\Omega) := W^{m,2}(\Omega)$ for each $m \in \mathbb{N}$. In this case, the norm reads

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

EXERCISE 3.3.8. Use Exercise 3.3.5 to show that the corresponding scalar product is given by

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}.$$

By introducing the gradient $\nabla u(\boldsymbol{x}) = (\partial_1 u(\boldsymbol{x}), \cdots, \partial_n u(\boldsymbol{x}))$, we see that

$$\begin{aligned} \int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} &= \int_\Omega \sum_{i=1}^n \partial_i u(\boldsymbol{x}) \partial_i v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\ &= \sum_{i=1}^n \int_\Omega \partial_i u(\boldsymbol{x}) \partial_i v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \sum_{i=1}^n (\partial_i u, \partial_i v)_{L^2(\Omega)} \end{aligned}$$

Therefore it is convenient to define

$$(\nabla u, \nabla v)_{L^2(\Omega)} := \int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \quad \|\nabla u\|_{L^2(\Omega)} := \sqrt{(\nabla u, \nabla u)_{L^2(\Omega)}},$$

so the scalar product and norm on $H^1(\Omega)$ can be expressed as

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}, \quad \|u\|_{H^1(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

We introduce the Hessian matrix $\nabla^{\otimes 2} u(\boldsymbol{x}) \equiv \nabla \otimes \nabla u(\boldsymbol{x})$ with entries $(\nabla^{\otimes 2} u(\boldsymbol{x}))_{ij} = \partial_i \partial_j u(\boldsymbol{x})$. The following simple exercise explains why we choose this notation.

EXERCISE 3.3.9. For each vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ (identify as $n \times 1$ matrix), we define the juxtaposition $\boldsymbol{a} \otimes \boldsymbol{b} \in \mathbb{R}^{n \times n}$ (i.e. an $n \times n$ matrix with entries in $\mathbb{R}$) by $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\intercal$, where $\intercal$ denotes the transpose of the vector. Compute each entry $(\boldsymbol{a} \otimes \boldsymbol{b})_{ij}$ of the $n \times n$ matrix $\boldsymbol{a} \otimes \boldsymbol{b}$.

EXERCISE 3.3.10. Let $\boldsymbol{e}_j \in \mathbb{R}^n$ be the j^{th} column of the identity matrix Id_n. Show that

$$\sum_{k=1}^n \boldsymbol{e}_k \otimes \boldsymbol{e}_k = \mathrm{Id}_n.$$

EXERCISE 3.3.11. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and consider the matrix $A := \mathrm{Id}_n + \boldsymbol{u} \otimes \boldsymbol{v}$, which is called the *rank-one perturbation of identity*. Determine the relation between $\boldsymbol{u}$ and $\boldsymbol{v}$ to guarantee A^{-1} exists, and compute A^{-1}.

We see that

$$\int_\Omega \nabla^{\otimes 2}u(\boldsymbol{x}) : \nabla^{\otimes 2}v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \sum_{i,j=1}^n \partial_i\partial_j u(\boldsymbol{x})\partial_i\partial_j v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$
$$= \sum_{i,j=1}^n \int_\Omega \partial_i\partial_j u(\boldsymbol{x})\partial_i\partial_j v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{i,j=1}^n (\partial_i\partial_j u, \partial_i\partial_j v)_{L^2(\Omega)}.$$

Therefore it is convenient to define

$$(\nabla^{\otimes 2}u, \nabla^{\otimes 2}v)_{L^2(\Omega)} := \int_\Omega \nabla^{\otimes 2}(\boldsymbol{x}) : \nabla^{\otimes 2}v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \|\nabla^{\otimes 2}u\|_{L^2(\Omega)} := \sqrt{(\nabla^{\otimes 2}u, \nabla^{\otimes 2}u)_{L^2(\Omega)}},$$

and

$$(u,v)_{H^2(\Omega)} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} + (\nabla^{\otimes 2}u, \nabla^{\otimes 2}v)_{L^2(\Omega)},$$
$$\|u\|_{H^2(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla^{\otimes 2}u\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \tag{3.3.2}$$

The $\|\cdot\|_{H^2(\Omega)}$-norm given in (3.3.2) is actually equivalent to the $\|\cdot\|_{H^2(\Omega)}$-norm given in Definition 3.3.7 in the following sense:

DEFINITION 3.3.12. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on the vector space X. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exists a constant $c > 0$ such that

$$c^{-1}\|u\|_1 \le \|u\|_2 \le c\|u\|_1 \quad \text{for all } u \in X.$$

Similarly, by introducing the k-tensor $\nabla^{\otimes k}u(\boldsymbol{x})$ with entries $\left(\nabla^{\otimes k}u(\boldsymbol{x})\right)_{i_1 i_2\cdots i_k} = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u(\boldsymbol{x})$, we see that

$$\int_\Omega \nabla^{\otimes k}u(\boldsymbol{x}) \overset{(k)}{\bullet} \nabla^{\otimes k}v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \sum_{i_1,\cdots,i_k=1}^n \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u(\boldsymbol{x})\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$
$$= \sum_{i_1,\cdots,i_k=1}^n \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u(\boldsymbol{x})\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$
$$= \sum_{i_1,\cdots,i_k=1}^n (\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u, \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}v)_{L^2(\Omega)}.$$

It is convenient to define $\nabla^{\otimes 0}u := u$, and for each $k \in \mathbb{N}$ that

$$(\nabla^{\otimes k}u, \nabla^{\otimes k}v)_{L^2(\Omega)} := \int_\Omega \nabla^{\otimes k}(\boldsymbol{x}) \overset{(k)}{\bullet} \nabla^{\otimes k}v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \|\nabla^{\otimes k}u\|_{L^2(\Omega)} := \sqrt{(\nabla^{\otimes k}u, \nabla^{\otimes k}u)_{L^2(\Omega)}}.$$

We define the scalar products and norm by

$$(u,v)_{H^m(\Omega)} = \sum_{k=0}^m (\nabla^{\otimes k}u, \nabla^{\otimes k}v)_{L^2(\Omega)}, \quad \|u\|_{H^m(\Omega)} = \left(\sum_{k=0}^m \|\nabla^{\otimes k}u\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \tag{3.3.3}$$

The $\|\cdot\|_{H^m(\Omega)}$-norm given in (3.3.3) is actually equivalent to the $\|\cdot\|_{H^2(\Omega)}$-norm given in Definition 3.3.7. We recall the following fact, which already mentioned in Theorem 3.2.8:

THEOREM 3.3.13 (Trace theorem). *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{0,1}$ boundary $\partial\Omega$, i.e. Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. The mapping*

$$\mathrm{Tr}\,: C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega), \quad \mathrm{Tr}\,(u) := u|_{\partial\Omega}$$

extends to a unique bounded linear surjective mapping $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, where $H^{\frac{1}{2}}(\partial\Omega) := \mathrm{Tr}\,(H^1(\Omega)) \subset L^2(\partial\Omega)$, which is a Hilbert space equipped with the quotient norm

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf_{u\in H^1(\Omega), \mathrm{Tr}\,(u)=g} \|u\|_{H^1(\Omega)}.$$

It is also possible to define the "traces" and "normal derivatives" on $\partial\Omega$ for H^m-functions (see Theorem 3.2.8), see also [**LM72**, Theorem 9.4] for similar results for higher order derivatives:

THEOREM 3.3.14 (Trace theorem, a special case of [**AH09**, Theorem 7.3.11]). *Let $m \in \mathbb{Z}_{\geq 2}$ and let Ω be a bounded $C^{m-1,1}$ domain in $\mathbb{R}^n$. The mapping*

$$u \in C^\infty(\overline{\Omega}) \mapsto (\,u|_{\partial\Omega}\,, \partial_{\boldsymbol{\nu}} u|_{\partial\Omega}) \in C^\infty(\partial\Omega) \times C^\infty(\partial\Omega),$$

where $\partial_{\boldsymbol{\nu}} u := \boldsymbol{\nu} \cdot \nabla u$, extends to a unique bounded linear surjective mapping $H^m(\Omega) \to H^{m-\frac{1}{2}}(\partial\Omega) \times H^{m-\frac{3}{2}}(\partial\Omega)$, where for each $k = 1, 2, \cdots, m$ the space $H^{k-\frac{1}{2}}(\partial\Omega) := \mathrm{Tr}\,(H^k(\Omega)) \subset L^2(\partial\Omega)$, which is a Hilbert space equipped with the quotient norm

$$\|g\|_{H^{k-\frac{1}{2}}(\partial\Omega)} = \inf_{u\in H^k(\Omega), \mathrm{Tr}\,(u)=g} \|u\|_{H^k(\Omega)}.$$

The following is an immediate corollary of Lemma 3.2.4:

COROLLARY 3.3.15 (Density). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $m \in \mathbb{N}$. Given any $f \in H^m(\Omega)$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ such that*

$$f_n \to f \text{ in } H^m(\Omega), \quad \text{that is,} \quad \lim_{k\to\infty} \|f_k - f\|_{H^m(\Omega)} = 0.$$

In view of Corollary 3.3.15, it is natural to consider the following subspace of $H^m(\Omega)$:

DEFINITION 3.3.16. For each $m \in \mathbb{N}$, we define $H_0^m(\Omega)$ be the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^m(\Omega)}$.

The relation between $H_0^m(\Omega)$ and $H^m(\Omega)$ are given in the followings:

LEMMA 3.3.17. *If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then*

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\right\}.$$

If Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, then

$$H_0^2(\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = \partial_{\boldsymbol{\nu}} u|_{\partial\Omega} = 0\right\}.$$

REMARK. The subscript 0 in $H_0^m(\Omega)$ means the zero boundary value. Therefore here we denote $C_c^\infty(\Omega)$ rather than $C_0^\infty(\Omega)$, which also used by many authors, to avoid confusion. Since $H_0^1(\Omega) \cap H^2(\Omega) = \{u \in H^2(\Omega) : u|_{\partial\Omega} = 0\}$ for any bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$, then $H_0^1(\Omega) \cap H^2(\Omega) \neq H_0^2(\Omega)$.

EXAMPLE 3.3.18. For each parameter $\omega \geq 0$, we consider the function $v(t, \boldsymbol{x}) = e^{\mathbf{i}\omega t} u(\boldsymbol{x})$. It is not difficult to see that

$$(\partial_t^2 - c^2\Delta) v(t, \boldsymbol{x}) = -c^2 e^{\mathbf{i}\omega t}(\Delta + k^2) u(\boldsymbol{x}) \quad \text{with } k = \frac{\omega}{c}.$$

Since $c \neq 0$ and $e^{\mathrm{i}\omega t} \neq 0$ for all $t \in \mathbb{R}$, the it is natural to consider the following second order *elliptic* PDE on a bounded Lipschitz domain Ω:

$$(\Delta + k^2)u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{3.3.4}$$

When $k = 0$, we call (3.3.4) the *Poisson equation*; when $k > 0$, we (3.3.4) the Helmholtz equation, which described the acoustic wave with fixed wave number $k > 0$ [**CCH23, CK19, KG08**]. The term $e^{\mathrm{i}\omega t}$ is called the time-harmonic, and thus we also called the Helmholtz equation the *time-harmonic wave equation*. In view of the integration by parts (Theorem 3.2.8), one first formally compute that

$$\begin{aligned}\int_\Omega f\phi \, \mathrm{d}\boldsymbol{x} &= \int_\Omega \Delta u \phi \, \mathrm{d}\boldsymbol{x} + k^2 \int_\Omega u\phi \, \mathrm{d}\boldsymbol{x} \\ &= -\int_\Omega \nabla u \cdot \nabla \phi \, \mathrm{d}\boldsymbol{x} + k^2 \int_\Omega u\phi \, \mathrm{d}\boldsymbol{x} \quad \text{for all } \phi \in C_c^\infty(\Omega).\end{aligned}$$

In view of the definition of $H_0^1(\Omega)$ and Lemma 3.3.17, we say that u is a weak solution of (3.3.4) if

$$u \in H_0^1(\Omega) \text{ and } (f,\phi)_{L^2(\Omega)} = -(\nabla u, \nabla \phi)_{L^2(\Omega)} + k^2 (u,\phi)_{L^2(\Omega)} \text{ for all } \phi \in H_0^1(\Omega) \tag{3.3.5}$$

for any pre-given $f \in L^2(\Omega)$.

One sees that $(f,\phi)_{L^2(\Omega)}$ is actually well-defined for $f,\phi \in L^2(\Omega)$, and here we have $\phi \in H_0^1(\Omega)$. Therefore it is natural to ask:

QUESTION 3.3.19. *Given any $\phi \in H_0^1(\Omega)$, whether the term $T_f(\phi) := (f,\phi)_{L^2(\Omega)}$ still make sense for lower regularity f?*

By using Exercise 1.0.9 and the density lemma (Lemma 1.0.15), we have

$$\|f\|_{L^2(\Omega)} = \sup_{0 \not\equiv \phi \in L^2(\Omega)} \frac{(f,\phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}} \equiv \sup_{0 \not\equiv \phi \in C_c^\infty(\Omega)} \frac{(f,\phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}}. \tag{3.3.6}$$

This is actually a special case of the following general notion:

DEFINITION 3.3.20. Let X and Y be two Banach spaces. An *unbounded linear operator* from X into Y is a linear map $\mathscr{L} : \mathrm{dom}\,(\mathscr{L}) \subset X \to Y$ defined on a linear spaces $\mathrm{dom}\,(\mathscr{L}) \subset X$ with values Y. The linear space $\mathrm{dom}\,(\mathscr{L})$ is called the *domain* of $\mathscr{L}$. If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, then we called such $\mathscr{L}$ a *linear functional* on the domain $\mathrm{dom}\,(\mathscr{L})$.

DEFINITION 3.3.21. One says that $\mathscr{L}$ is *bounded* (or *continuous*) if $\mathrm{dom}\,(\mathscr{L}) = X$ and if there is a constant $c \geq 0$ such that

$$\|\mathscr{L}u\| \leq c\|u\| \quad \text{for all } u \in X.$$

The norm of a bounded operator is defined by

$$\|\mathscr{L}\|_{X\to Y} := \inf\{c \geq 0 : \|\mathscr{L}u\| \leq c\|u\| \text{ for all } u \in X\} \equiv \sup_{u \neq 0} \frac{\|\mathscr{L}u\|_Y}{\|u\|_X}.$$

If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, one says that such $\mathscr{L}$ is a *bounded* (or *continuous*) *linear functional* on X.

EXERCISE 3.3.22. Verify that $\|\cdot\|_{X\to Y}$ given in Definition 3.3.21 is a norm.

Similar to distributions (Definition 3.1.19), we now introduce the following definition:

DEFINITION 3.3.23. Let H be a Hilbert space. The *dual space* H^* of H is a Hilbert space consists of all bounded linear functional on H, with norm $\|\cdot\|_{H^*} = \|\cdot\|_{H\to\mathbb{R}}$.

We now consider a trick similar to Example 3.1.17:

EXAMPLE 3.3.24. The equality (3.3.6) means that f can be identify with the bounded linear functional $T_f(\phi) := (f,\phi)_{L^2(\Omega)}$ for all $\phi \in L^2(\Omega)$. In other words, we have $L^2(\Omega) = (L^2(\Omega))^*$.

We now ready to answer Question 3.3.19. Since we have $\phi \in H_0^1(\Omega)$, together with the density lemma (Corollary 3.3.15), the equation (3.3.6) suggests us to define the following quantity:

$$\|f\|_{H^{-1}(\Omega)} := \sup_{\phi\not\equiv 0} \frac{\int_\Omega f\phi \,\mathrm{d}\boldsymbol{x}}{\|\phi\|_{H_0^1(\Omega)}} \equiv \sup_{\|\phi\|_{H_0^1(\Omega)}=1} \int_\Omega f\phi\,\mathrm{d}\boldsymbol{x}$$

Here we write $\int_\Omega f\phi\,\mathrm{d}\boldsymbol{x}$ rather than $(f,\phi)_{L^2(\Omega)}$ because here f may not in $L^2(\Omega)$. We see that $\|f\|_{H^{-1}(\Omega)} = \|T_f\|_{(H_0^1(\Omega))^*}$, therefore we immediately can define $H^{-1}(\Omega) := (H_0^1(\Omega))^*$, more precisely:

DEFINITION 3.3.25. For each bounded Lipschitz domain Ω in $\mathbb{R}^n$, we define

$$H^{-1}(\Omega) := \left\{ f \in \mathscr{D}'(\Omega) : T_f \in (H_0^1(\Omega))^* \right\}.$$

From the above discussions, we obtain the triplet

$$H_0^1(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))^* \subset H^{-1}(\Omega). \tag{3.3.7}$$

From definition, one can easily note that

$$\left| \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right| \le \|f\|_{H^{-1}(\Omega)} \|g\|_{H_0^1(\Omega)}.$$

Warning. However, one should aware that in general

$$\int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \text{ cannot be bounded from above by } \|f\|_{H^{-1}(\Omega)}\|g\|_{H_0^1(\Omega)},$$

it is interesting to compare this with (1.0.4) in Exercise 1.0.9.

REMARK 3.3.26. Similarly, for each bounded smooth (for simplicity) domain Ω in $\mathbb{R}^n$, one can define $H^{-m}(\Omega) := (H_0^m(\Omega))^*$ for each $m \in \mathbb{N}$, more precisely,

$$H^{-m}(\Omega) := \{ f \in \mathscr{D}'(\Omega) : T_f \in (H_0^m(\Omega))^* \}.$$

Similarly, we also obtain the triplet

$$H_0^m(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))^* \subset H^{-m}(\Omega) \quad \text{for all } m \in \mathbb{N}. \tag{3.3.8}$$

This ideas also can be extended for real numbers $m \ge 0$ and bounded Lipschitz domains [**McL00**], see also [**LM72**]. Here we also remark that $(H^m(\Omega))^*$ for (real) $m \ge 0$ can be characterized in terms of quotient norm [**McL00**].

We now exhibit the following remarkable fact for all bounded linear functionals on Hilbert spaces:

THEOREM 3.3.27 (Riesz-Fréchet representation theorem [**Bre11**, Theorem 5.5]). *Given any $\varphi \in H^*$, there exists a unique $v \in H$ with $\|v\|_H = \|\varphi\|_{H^*}$ such that*

$$\varphi(u) = (v,u)_H \text{ for all } u \in H.$$

REMARK. Due to the above Riesz-Fréchet theorem, we also denote $\varphi(u) = \langle\varphi, u\rangle$, or more precisely, $\varphi(u) = \langle\varphi, u\rangle_{H^*\otimes H}$, and we call $\langle\cdot,\cdot\rangle$ the *duality pair*.

The Riesz-Fréchet representation theorem (Theorem 3.3.27) defines the isometry $\iota : H^* \to H$ by $\iota(\varphi) := v$. Similar to Sobolev embeddings (Theorem 3.2.6), this suggests us to identify H and H^*, but this sometimes cause some troubles. For example, if we identify both $L^2(\Omega) \cong (L^2(\Omega))^*$ and $H_0^1(\Omega) \cong (H_0^1(\Omega))^* = H^{-1}(\Omega)$, then the triplet (3.3.7) implies $L^2 \cong H_0^1(\Omega) \cong H^{-1}(\Omega)$, which obviously make no sense. In typical situation, we usually identify $L^2(\Omega) = (L^2(\Omega))^*$ and not identify $H_0^1(\Omega)$ with its dual $H^{-1}(\Omega)$ despite we have the Riesz-Fréchet representation theorem (Theorem 3.3.27).

3.4. Solving elliptic PDE for small wave number

We now turn back to the Helmholtz equation (3.3.4), and we now can ask the following question (in a proper way):

QUESTION 3.4.1 (See also Question 3.6.21 for a slightly general case). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, let $f \in H^{-1}(\Omega)$ and $k \geq 0$, can we find a weak solution u of the Helmholtz equation (3.3.4)? More precisely, can we find $u \in H_0^1(\Omega)$ satisfies*

$$(\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u,v)_{L^2(\Omega)} = -\langle f, v\rangle_{H^{-1}\otimes H_0^1(\Omega)} \text{ for all } v \in H_0^1(\Omega)$$

or not? In addition, is the solution unique?

In view of the above formulation, it is natural to introduce the following notions:

DEFINITION 3.4.2. A bilinear form $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ (see Definition 3.3.1) is said to be

(a) *continuous* if there is a constant $C > 0$ such that
$$|a(u,v)| \leq C\|u\|_H\|v\|_H \quad \text{for all } u,v \in H.$$
(b) *coercive* if there is a constant $\alpha > 0$ such that
$$a(v,v) \geq \alpha\|v\|_H^2 \quad \text{for all } v \in H.$$
(c) *symmetric* if $a(u,v) = a(v,u)$ for all $u,v \in H$. In this case, the coercive bilinear form a is also said to be *positive definite*.

REMARK 3.4.3 (Finite dimensional case: linear algebra). We say that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if

$$Au \cdot u = u^\intercal Au > 0 \quad \text{for all } 0 \neq u \in \mathbb{R}^n. \tag{3.4.1}$$

Since A is symmetric, the it is unitary diagonalizable (iff A is normal, i.e. $AA^\intercal = A^\intercal A$), i.e. there exists an invertible matrix Q with $Q^{-1} = Q^\intercal$ such that $A = QDQ^T$, where D is a diagonal matrix. Hence (3.4.1) writes

$$u^\intercal QDQ^\intercal u = D(Q^\intercal u)\cdot(Q^\intercal u) = D(Q^{-1}u)\cdot(Q^{-1}u) > 0 \quad \text{for all } 0 \neq u \in \mathbb{R}^n,$$

this equivalent to

$$Dv \cdot v > 0 \quad \text{for all } 0 \neq v \in \mathbb{R}^n.$$

Therefore, all entries in D, they called the *eigenvalue* of A, must be positve. This explains why we the condition (3.4.1) called positive definite, and so is the above definition for infinite dimensional case.

We now exhibit the following remarkable result, which is a very simple and efficient tool for solving linear elliptic PDE:

THEOREM 3.4.4 (Lax-Milgram [**Bre11**, Corollary 5.8]). *Assume that $a(\cdot,\cdot)$ is continuous coercive bilinear form on H. Then, given any $\varphi \in H^*$, there exists a unique element $u \in H$ such that*

$$a(u,v) = \langle \varphi, v\rangle \quad \text{for all } v \in H. \tag{3.4.2}$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in H, \quad \frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in H}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}. \tag{3.4.3}$$

REMARK 3.4.5. In the language of the calculus of variations, one says that (3.4.2) the *Euler equation* associated with the minimization problem (3.4.3).

In order to answer Question 3.4.1, it is now natural to consider

$$a(u,v) := (\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u,v)_{L^2(\Omega)}, \quad \varphi = -f \in H^{-1}(\Omega). \tag{3.4.4}$$

It is easy to verify that $a : H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ is a continuous and symmetric bilinear form. In order to verify its coercivity, we need the following lemma:

LEMMA 3.4.6 (Poincaré's inequality [**Bre11**, Corollary 9.19]). *Let Ω be a bounded open set. Then there exists a constant C, depending on Ω, such that*

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega).$$

EXERCISE 3.4.7. Let I be a bounded interval in $\mathbb{R}$. Show that there exists a constant C, depending on the length of the interval $|I| < \infty$, such that

$$\|u\|_{L^2(I)} \le C\|u'\|_{L^2(I)} \quad \text{for all } u \in H_0^1(I).$$

(Hint: This result is not optimal, you will see the optimal inequality in the proof.)

By using the Poincaré inequality and the density result (Corollary 3.3.15), one can define a positive number, called the *fundamental tone* of Ω, by

$$\lambda_1 := \inf_{0\not\equiv u \in C_c^\infty(\Omega)} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} = \inf_{0\not\equiv u\in H_0^1(\Omega)} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} > 0. \tag{3.4.5}$$

The quotient $\|\nabla u\|_{L^2(\Omega)}^2/\|u\|_{L^2(\Omega)}^2$ sometimes also referred as the *Rayleigh quotient*. Hence

$$a(u,u) = \|\nabla u\|_{L^2(\Omega)}^2 - k^2\|u\|_{L^2(\Omega)}^2 \ge \left(1 - \frac{k^2}{\lambda_1}\right)\|\nabla u\|_{L^2(\Omega)}^2,$$

which means that a is coercive when $k^2 < \lambda_1$. By using the Lax-Milgram theorem (Theorem 3.4.4), we can give some partial answers to Question 3.4.1:

THEOREM 3.4.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $f \in H^{-1}(\Omega)$. If $k^2 < \lambda_1$, then there exists a unique $u \in H_0^1(\Omega)$ satisfies*

$$(\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u,v)_{L^2(\Omega)} = -\langle f, v\rangle_{H^{-1}\otimes H_0^1(\Omega)} \text{ for all } v \in H_0^1(\Omega)$$

satisfying

$$F(u) = \min_{v\in H_0^1(\Omega)} F(v), \quad F(v) = \frac{1}{2}\left(\|\nabla v\|_{L^2(\Omega)}^2 - k^2\|v\|_{L^2(\Omega)}^2\right) + \langle f, v\rangle_{H^{-1}(\Omega)\otimes H_0^1(\Omega)}.$$

3.5. The maximum principle

We now investigate the solution u of the Helmholtz equation (3.3.4), in the sense of Theorem 3.4.8. In order to make our statement make sense, we first introduce the following notion.

DEFINITION 3.5.1 ([**KS00**, Definition 5.1]). Let Ω be an open set in $\mathbb{R}^n$, let $u \in H^1(\Omega)$ and let $E \subset \overline{\Omega}$. We say that the function u is *nonnegative* on E in the sense of $H^1(\Omega)$, or we simply denote $u \geq 0$ on E in $H^1(\Omega)$, if there exists a sequence $u_n \in H^{1,\infty}(\Omega)$ such that

$$u_n(x) \geq 0 \text{ for all } x \in E, \quad u_n \to u \text{ in } H^1(\Omega).$$

If $-u \geq 0$ on E in $H^1(\Omega)$, then u is *nonpositive* on E in $H^1(\Omega)$,or we simply denote $u \leq 0$ on E in $H^1(\Omega)$. Similarly, we say that $u \leq v$ on E in $H^1(\Omega)$ if $v - u \geq 0$ on E in $H^1(\Omega)$. Accordingly, we also define

$$\sup_E u := \inf \left\{ M \in \mathbb{R} : u \leq M \text{ on } E \text{ in } H^1(\Omega) \right\}.$$

In fact, the following density lemma **with signed constraint** (which does not mentioned in Corollary 3.3.15) guarantees that the notion of $\geq$ in $H^1(\Omega)$ is consistent with the notion of $\geq$ in the Lebesgue measure sense.

LEMMA 3.5.2 ([**KS00**, Proposition 5.2]). *Let Ω be an open set in $\mathbb{R}^n$, let $u \in H^1(\Omega)$ and let $E \subset \overline{\Omega}$.*

(i) *If $u \geq 0$ on E in $H^1(\Omega)$, then $u \geq 0$ on E a.e.*
(ii) *If $u \geq 0$ on Ω a.e., then $u \geq 0$ on Ω in $H^1(\Omega)$.*
(iii) *If $u \in H_0^1(\Omega)$ and $u \geq 0$ on Ω a.e., then there exists a sequence $u_n \in W_0^{1,\infty}(\Omega)$ such that $u_n \geq 0$ in Ω and $u_n \to u$ in $H_0^1(\Omega)$.*
(iv) *If E is open in Ω and $u \geq 0$ on E a.e., then for each compact set $K \subset E$, one has $u \geq 0$ on E in $H^1(\Omega)$.*

(i) and (ii) implies $u \geq 0$ on Ω a.e. if and only if $u \geq 0$ on Ω in $H^1(\Omega)$. Hence, from now on, we can just simply write $u \geq 0$ in Ω for all $u \in H^1(\Omega)$. The following lemma is also helpful:

LEMMA 3.5.3 ([**AK19**, Lemma 2.5][1]). *Let Ω be an open set in $\mathbb{R}^n$ and let $u \in H^1(\Omega)$. We set $u_+(x) := \max\{u(x), 0\}$ for a.e. $x \in \Omega$. Then $u_+ \in H^1(\Omega)$ and $\mathrm{Tr}\,(u_+) = (\mathrm{Tr}\, u)_+$ for $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$ with*

$$\nabla(u_+) = \begin{cases} \nabla u & \text{a.e. in } \Omega_+ := \{x \in \Omega : u(x) > 0\}, \\ 0 & \text{a.e. in } \Omega \setminus \Omega_+. \end{cases} \tag{3.5.1}$$

EXERCISE 3.5.4. Verify (3.5.1) by assuming that $u \in H^1(\Omega)$ and $u_+ \in H^1(\Omega)$.

We now ready to prove the following lemma (it is interesting to compare the ideas in the following lemma with [**GT01**, Theorem 8.1]).

LEMMA 3.5.5 (Weak maximum principle [**KLSS22**, Proposition A.5 in arXiv version]). *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $\lambda_1(\Omega)$ be the fundamental tone of Ω given in (3.4.5). If $k^2 < \lambda_1(\Omega)$, $u \in H^1(\Omega)$ satisfies*

(a) *$u|_{\partial\Omega} \leq 0$ in the sense of $u_+ \in H_0^1(\Omega)$, which is well-defined by Lemma 3.5.3 above;*

[1] see also [**KS00**, Theorem A.1]

(b) $-(\Delta + k^2)u \leq 0$ *in the sense of* $H^{-1}(\Omega)$*, i.e.* $a(u,v) \leq 0$ *for all* $v \in H_0^1(\Omega)$ *with* $v \geq 0$*, where* a *is the bilinear form given in* (3.4.4)*;*

then $u \leq 0$ *in* Ω*.* (*one can refer to Remark 2.3.5 for the minus sign in* (b))

PROOF. By using Lemma 3.5.3, we observe that $(\nabla u_+, \nabla u_-)_{L^2(\Omega)} = 0$, therefore from (a) and (b) one sees that

$$\|\nabla u_+\|_{L^2(\Omega)}^2 = (\nabla u, \nabla u_+)_{L^2(\Omega)} \leq k^2 \|u_+\|_{L^2(\Omega)}^2.$$

From (a), we have $u_+ \in H_0^1(\Omega)$, therefore Poincaré inequality (Lemma 3.4.6) gives

$$\|u_+\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_1(\Omega)} \|\nabla u_+\|_{L^2(\Omega)}^2.$$

Combining the above two inequalities, we reach

$$\|u_+\|_{L^2(\Omega)}^2 \leq \frac{k^2}{\lambda_1(\Omega)} \|u_+\|_{L^2(\Omega)}^2.$$

Since $\frac{k^2}{\lambda_1(\Omega)} < 1$, it follows that $w_+ \equiv 0$, and therefore $w \leq 0$ in Ω. □

Indeed, by using the mean value theorem for Helmholtz operator, one can show the following lemma (we will not show the proof here, also compare this with [**GT01**, Theorem 8.19]).

LEMMA 3.5.6 (Strong maximum principle [**KLSS22**, Proposition A.5 in arXiv version]). *Suppose that all assumptions in Lemma 3.5.5 hold. If we additionally assume that* u *is continuous in* Ω*, then in each connected component of* Ω *we have either* $u < 0$ *or* $u \equiv 0$*.*

Finally, we closed this section by remark that it is also possible to formula the maximum principle for *arbitrary* open set Ω (without any regularity assumption on its boundary $\partial\Omega$), see [**BNV94**].

3.6. Solving elliptic PDE: Eigenvalue problem and Fredholm alternative

We now turn back to Question 3.4.1. Since we confirmed the case $k^2 < \lambda_1$ in Theorem 3.4.8, then it is natural to ask whats going for the case when $k^2 = \lambda_1(\Omega)$. Under some suitable assumptions on Ω, later we will show that the infimum in (3.4.5) can be achieved, i.e. one can find $0 \not\equiv u_* \in H_0^1(\Omega)$ such that

$$\lambda_1 = \inf_{0 \not\equiv u \in H_0^1(\Omega)} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} = \frac{\|\nabla u_*\|_{L^2(\Omega)}^2}{\|u_*\|_{L^2(\Omega)}^2}, \tag{3.6.1}$$

i.e. $\|\nabla u_*\|_{L^2(\Omega)}^2 = \lambda_1 \|u_*\|_{L^2(\Omega)}^2$, which suggests that u_* may satisfies $(\Delta + \lambda_1)u_* = 0$ and $u_*|_{\partial\Omega} = 0$, therefore we have to expect a negative answer (the solution is not unique) for Question 3.4.1 when $k^2 = \lambda_1$.

DEFINITION 3.6.1. Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and the norm $\|\cdot\|_H$. We say that a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ is a *Hilbert basis* or *orthonormal basis* of H if it satisfies the following properties:

(1) $\|\phi_k\|_H = 1$ for all $k \in \mathbb{N}$ and $(\phi_k, \phi_j)_H = 0$ for all $k \neq j$.

(2) the linear space spanned by $\{\phi_k\}_{k\in\mathbb{N}}$, i.e.

$$\operatorname{span}\{\phi_k\}_{k\in\mathbb{N}} := \left\{ \sum_{k\in I} c_k\phi_k : I \text{ is a finite set in } \mathbb{N}, c_k \in \mathbb{R} \right\},$$

is dense in H.

We exhibit the following remarkable fact about Hilbert basis (here we do not show the proof).

THEOREM 3.6.2. *Let H be a separable (i.e. there exists a subset $X \subset H$ which is countable and dense) Hilbert space and let $\{\phi_k\}_{k\in\mathbb{N}}$ be an orthonormal subset of H. Then the following are equivalent:*

(1) *$\{\phi_k\}_{k\in\mathbb{N}}$ is a Hilbert basis;*
(2) *The following Parseval identity holds:*

$$\|f\|_H^2 = \sum_{k\in\mathbb{N}} |(f,\phi_k)_H|^2;$$

(3) *If $f \in H$ and $(f,\phi_k)_H = 0$ for all $k \in \mathbb{N}$, then $f \equiv 0$.*

COROLLARY 3.6.3. *Let H be a separable Hilbert space with Hilbert basis $\{\phi_k\}_{k\in\mathbb{N}}$. For each $u \in H$, one has*

$$u = \sum_{k=1}^{\infty} (u,\phi_k)_H \phi_k \text{ converges in } H. \tag{3.6.2}$$

The precise meaning of (3.6.2) is

$$\lim_{m\to\infty} \left\| u - \sum_{k=1}^{m} (u,\phi_k)_H\phi_k \right\|_H = 0.$$

In fact, we have the following well-known result:

THEOREM 3.6.4 (Spectral decomposition of Dirichlet Laplacian [**Bre11**, Corollary 9.19]). *Let Ω be a bounded Lipschitz domain. There exists a Hilbert basis $\{\phi_j\}_{j\in\mathbb{N}}$ of $L^2(\Omega)$ and a sequence of real numbers $\{\lambda_j\}_{j\in\mathbb{N}}$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ such that*

$$\phi_j \in H_0^1(\Omega) \cap C^\infty(\Omega), \quad -\Delta\phi_j = \lambda_j\phi_j \text{ in } \Omega.$$

We usually call $\{\lambda_j\}_{j\in\mathbb{N}}$ the eigenvalues *of Dirichlet Laplacian on Ω, and $\{\phi_j\}_{j\in\mathbb{N}}$ be their corresponding* eigenfunctions. *Sometimes we call $\{(\lambda_j,\phi_j)\}_{j\in\mathbb{N}}$ the eigensystem of Dirichlet Laplacian on Ω. (one can refer to Remark 2.3.5 for the minus sign in elliptic operators)*

This immediately gives negative results for Question 3.4.1 for $k^2 = \lambda_j$ for some $j \in \mathbb{N}$: the solution of Helmholtz equation (3.3.4) is not unique when $k^2 = \lambda_j$ for some $j \in \mathbb{N}$. We will not going to give a detailed proof of the eigendecomposition of Dirichlet Laplacian (Theorem 3.6.4). As an introduction, we slightly mention that the proof of Theorem 3.6.4 involving the following notion:

DEFINITION 3.6.5. Let X and Y be two Banach spaces. A bounded linear operator $T : X \to Y$ is said to be compact if for any bounded sequence $\{x_j\}_{j\in\mathbb{N}}$ in X, the sequence $\{Tx_j\}_{j\in\mathbb{N}}$ contains a convergent subsequence in Y. The set of all compact operators from X to Y is denoted as $\mathcal{K}(X,Y)$. If $X = Y$, we simply write $\mathcal{K}(X) = \mathcal{K}(X,X)$.

The following fact is useful to check whether the bounded linear operator is compact or not.

LEMMA 3.6.6 ([**Bre11**, Proposition 6.3]). *Let X, Y and Z be three Banach spaces. Let $\mathcal{L}(X,Y)$ denotes the set of bounded linear operator from X to Y.*

(1) *If $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{K}(Y,Z)$, then $S \circ T \in \mathcal{K}(X,Z)$.*
(2) *If $T \in \mathcal{K}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$, then $S \circ T \in \mathcal{K}(X,Z)$.*

We usually use the following form of Lemma 3.6.6: If $Y \Subset Z$, i.e. the embedding $\iota : Y \to Z$ is compact, then by identifying $T \cong \iota \circ T$, one sees that $T \in \mathcal{L}(X,Y)$ implies $T \in \mathcal{K}(X,Z)$. Similarly, if $X \Subset Y$, one sees that $S \in \mathcal{L}(Y,Z)$ implies $S \in \mathcal{K}(X,Z)$.

REMARK 3.6.7. By using the Poincaré inequality (Lemma 3.4.6), one sees that $H_0^1(\Omega)$ is a Hilbert space equipped with the scalar product $(u,v)_{H_0^1(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)}$. From the weak formulation of $-\Delta\phi_j = \lambda_j\phi_j$ in Ω, together with the density lemma (Corollary 3.3.15), one sees that

$$\sum_{k=1}^{m}(u,\phi_k)_{H_0^1(\Omega)}\phi_k = \sum_{k=1}^{m}(\nabla u, \nabla\phi_k)_{L^2(\Omega)}\phi_k = \sum_{k=1}^{m}\lambda_k(u,\phi_k)_{L^2(\Omega)}\phi_k,$$

and then from Theorem 3.6.2 we see that

$$\|u\|_{H_0^1(\Omega)}^2 = \sum_{k=1}^{\infty}|(u,\phi_k)_{H_0^1(\Omega)}|^2 = \sum_{k=1}^{\infty}\lambda_k^2|(u,\phi_k)_{L^2(\Omega)}|^2. \tag{3.6.3}$$

From this, one then easily see that

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|u\|_{H_0^1(\Omega)}^2 = \sum_{k=1}^{\infty}(1+\lambda_k^2)|(u,\phi_k)_{L^2(\Omega)}|^2.$$

REMARK 3.6.8. From (3.6.3), in fact one can define the fractional order Sobolev by

$$\|u\|_{H_0^s(\Omega)}^2 := \sum_{k=1}^{\infty}\lambda_k^{2s}|(u,\phi_k)_{L^2(\Omega)}|^2 \quad \text{for } s \in (0,1),$$

with scalar product

$$(u,v)_{H_0^s(\Omega)} := \sum_{k=1}^{\infty}\lambda_k^{2s}(u,\phi_k)_{L^2(\Omega)}(v,\phi_k)_{L^2(\Omega)} \quad \text{for } s \in (0,1).$$

For each $u \in H_0^s(\Omega)$, one also may define the spectral factional Laplacian (not to confused with Fourier fractional Laplacian) $(-\Delta)^s u \in H^{-s}(\Omega) := (H_0^s(\Omega))^*$ by

$$\langle(-\Delta)^s u, v\rangle_{H^{-s}(\Omega)\oplus H_0^s(\Omega)} := (u,v)_{H_0^s(\Omega)} \quad \text{for all } v \in H_0^s(\Omega).$$

However, the above idea cannot directly extend to arbitrary $s \in \mathbb{R}$, which require additionally assumptions on the boundary on u to make sure the definition is consistent with the classical Laplacian, see e.g. my dissertation [**Kow21**], which is indeed a lecture note, for further details.

It is worth-mentioning the following properties:

THEOREM 3.6.9 ([**GT01**, Theorem 8.38]). *Let Ω be a bounded Lipschitz domain, and let λ_1 be the first eigenvalue (also called the* principal eigenvalue*) given in the eigendecomposition in Theorem 3.6.4. Then the corresponding eigenfunction is simple, that is, if ϕ_1 and $\tilde{\phi}_1$ are both eigenfunctions corresponding to λ_1, then there exists a constant $c \neq 0$ such that $\phi_1(x) = c\tilde{\phi}_1(x)$ for all $x \in \Omega$. In addition, one can choose ϕ_1 such that $\phi_1(x) > 0$ for all $x \in \Omega$.*

REMARK. If we further assuming that $\partial\Omega$ is $C^{2,\alpha}$ for some $0 < \alpha < 1$, as a consequence of Krein-Rutman theorem, one has $\phi_1 \in C^{1,\alpha}(\overline{\Omega})$, see e.g. [**Du06**, Theorem 1.3].

We now turn back to answer (3.6.1), i.e. whether the infimum in Rayleigh quotient (3.4.5) can be achieved or not. In order to answer this question, need the following facts on compact operators, which can be found in [**dF82, dFG92**] as well as my dissertation [**Kow21**, Appendix A.5]:

LEMMA 3.6.10. *Let H be a Hilbert space and let $T \in \mathcal{K}(H)$ which is symmetric, i.e. $(Tu,v)_H = (u,Tv)_H$ for all $u,v \in H$. If*

$$\mu_1 = \sup\left\{(Tu,u)_H : \|u\|_H = 1\right\},$$

then there exists $\phi_1 \in H$ with $\|\phi_1\|_H = 1$ such that

$$(T\phi_1,\phi_1) = \mu_1, \quad T\phi_1 = \mu_1\phi_1.$$

Inductively for $j \geq 2$, if

$$\mu_j = \sup\left\{(Tu,u)_H : \|u\|_H = 1, u \perp \phi_i \text{ for all } i = 1,\cdots,j-1\right\},$$

where $u \perp v$ means $(u,v)_H = 0$, then there exists $\phi_j \in H$ with $\|\phi_j\|_H = 1$ with $\phi_j \perp \phi_1,\cdots,\phi_{j-1}$ such that

$$(T\phi_j,\phi_j) = \mu_j, \quad T\phi_j = \mu_j\phi_j.$$

EXERCISE 3.6.11. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Given $f \in H^{-1}(\Omega)$, by using Theorem 3.4.8 with $k=0$ one can find a unique solution $u \in H_0^1(\Omega)$ of the problem

$$(\nabla u,\nabla v)_{L^2(\Omega)} = \langle f,v\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)} \quad \text{for all } v \in H_0^1(\Omega). \tag{3.6.4}$$

Then one can define the linear operator $(-\Delta)_{\rm Dir}^{-1}f := u$. By doing some suitable identifications, show that $(-\Delta)_{\rm Dir}^{-1} \in \mathcal{K}(L^2(\Omega))$.

EXERCISE 3.6.12 (The equivalence of first Dirichlet eigenvalue and fundamental tone). By choosing $T = (-\Delta)_{\rm Dir}^{-1}$ and $H = L^2(\Omega)$, where $(-\Delta)_{\rm Dir}^{-1}$ is the operator mentioned in Exercise 3.6.11, show that the infimum in Rayleigh quotient (3.4.5) can be achieved, i.e. there exists $u_* \in H_0^1(\Omega)$ such that

$$\inf_{0\not\equiv u\in H_0^1(\Omega)} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} = \frac{\|\nabla u_*\|_{L^2(\Omega)}^2}{\|u_*\|_{L^2(\Omega)}^2}.$$

In addition, show that there exists a constant $c \neq 0$ such that $u_* = c\phi_1$.

EXERCISE 3.6.13 (Rayleigh quotient). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and we consider the eigendecomposition in Theorem 3.6.4. Show that

$$\lambda_j = \min\left\{\frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} : 0 \not\equiv u \in H_0^1(\Omega) \text{ with } u \perp \phi_i \text{ for all } i = 1,\cdots,j-1\right\}.$$

Here $u \perp \phi_i$ means $(u,\phi_i)_{L^2(\Omega)} = 0$.

The following Courant minimax principle [**CH04a**] (see [**CH04b**] for volume 2), which also can be found in [**dF82, dFG92**] as well as my dissertation [**Kow21**, Appendix A.5], enables us to characterize the eigenvalues without referring previous eigenvalues:

LEMMA 3.6.14. *Suppose that all assumptions in Lemma 3.6.10 hold. Then*

$$\mu_j = \max_{F_j \subset H, \dim(F_j)=j} \left(\inf \{ (Tu,u)_H : \|u\|_H = 1, u \in F_j \} \right),$$

where $\max\limits_{F_j \subset H, \dim(F_j)=j}$ *means the maximum taken over all subspaces* F_j *of* H *with* $\dim(F_j) = j$.

EXERCISE 3.6.15 (Courant minimax principle). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and we consider the eigendecomposition in Theorem 3.6.4. Show that

$$\lambda_j = \min_{F_j \subset H_0^1(\Omega), \dim F_j = j} \left(\max_{u \in F_j} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} \right),$$

where $\min\limits_{F_j \subset H_0^1(\Omega), \dim F_j = j}$ means that the minimum is taken over all finite dimensional vector space $F_j \subset H_0^1(\Omega)$ with $\dim F_j = j$.

REMARK 3.6.16. If we replace $H_0^1(\Omega)$ by $H^1(\Omega)$ in the Rayleigh quotient (3.4.5), or more generally the Courant minimax principle above, then this produces Neumann eigenfunctions for Laplacian. Since this is a bit technical, here we will not explain in this lecture note.

There is also another characterization for principal eigenvalue:

THEOREM 3.6.17 (see e.g. [**BNV94**]). *Let* Ω *be a bounded Lipschitz domain, and let* λ_1 *be the first eigenvalue (also called the* principal eigenvalue*) given in the eigendecomposition in Theorem 3.6.4. Then*

$$\lambda_1 = \max \left\{ \lambda \in \mathbb{R} : \textit{there exists a function } 0 \le u \in H^1(\Omega) \textit{ such that } (\Delta + \lambda)u \le 0 \textit{ in } \Omega \right\}.$$

Here $(\Delta+\lambda)u \le 0$ *in* Ω *means that* $(\nabla u \cdot \nabla v)_{L^2(\Omega)} - \lambda(u,v)_{L^2(\Omega)} \ge 0$ *for all* $0 \le v \in C_c^\infty(\Omega)$.

We now begin discuss the case when $k^2 \neq \lambda_j$ for all $j \in \mathbb{N}$ in the following exercise:

EXERCISE 3.6.18. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $u \in H_0^1(\Omega)$. If $k^2 \neq \lambda_j$ for all $j \in \mathbb{N}$, show that

$$u \equiv 0 \text{ in } \Omega \quad \text{if and only if} \quad (\Delta + k^2)u = 0 \text{ in } \Omega \text{ (give a suitable formulation).}$$

We will need the following theorem, which is a consequence of the Fredholm alternative [**Bre11**, Theorem 6.6]:

THEOREM 3.6.19 (Uniqueness implies existence result for Fredholm operators). *Let* X *be a Banach space, let* $f \in X$ *and let* $T \in \mathcal{K}(X)$. *If there is at most one solution of the equation* $(\mathrm{Id} - T)u = f$, *then there* exists *a unique solution* u *of* $(\mathrm{Id} - T)u = f$. *More precisely, if*

$$v = 0 \textit{ iff } v - Tv = 0 \quad \textit{i.e. } \mathrm{Id} - T \textit{ is injective,}$$

then for each $f \in X$ *there* exists *a unique solution* $u \in X$ *such that* $(\mathrm{Id} - T)u = f$.

REMARK. In fact, $\mathrm{Id} - T$ is a Fredholm operator of index zero.

Given any $f \in H^{-1}(\Omega)$, we notice that finding a $H_0^1(\Omega)$ solution of the Helmholtz equation (3.3.4) is equivalent to find a solution $u \in H_0^1(\Omega)$ of the following equation:

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \langle k^2 u - f, v\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)} \quad \text{for all } v \in H_0^1(\Omega).$$

If we consider the compact operator $(-\Delta)_{\rm Dir}^{-1}$ given in (3.6.4), the above equation means

$$u = (-\Delta)_{\rm Dir}^{-1}(k^2 u - f) \equiv k^2(-\Delta)_{\rm Dir}^{-1}u - (-\Delta)_{\rm Dir}^{-1}f,$$

equivalently,

$$(\mathrm{Id} - k^2(-\Delta)_{\rm Dir}^{-1})u = -(-\Delta)_{\rm Dir}^{-1}f.$$

By Exercise 3.6.11, it is easy to see that $T := k^2(-\Delta)_{\rm Dir}^{-1} \in \mathcal{K}(L^2(\Omega))$. Finally, we combine Exercise 3.6.18 and the Fredholm alternative (Theorem 3.6.19) to conclude the following theorem:

THEOREM 3.6.20 (See also Theorem 3.4.8 for refinement when $k^2 < \lambda_1$). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $f \in H^{-1}(\Omega)$. If $k^2 \neq \lambda_j$ for all $j \in \mathbb{N}$, then there exists a unique $u \in H_0^1(\Omega)$ satisfies*

$$(\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u,v)_{L^2(\Omega)} = -\langle f, v\rangle_{H^{-1}\oplus H_0^1(\Omega)} \text{ for all } v \in H_0^1(\Omega).$$

We now give a somehow satisfying answer to Question 3.4.1 (here we will not explain how the weak solution related to strong solution). We now ask another question which is slightly general than Question 3.4.1:

QUESTION 3.6.21. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, let $f \in H^{-1}(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $k \geq 0$ with k^2 is not an eigenvalue as mentioned in Theorem 3.6.4, can we find a weak solution u of the Helmholtz equation*

$$(\Delta + k^2)u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = g \tag{3.6.5}$$

or not? (Note: the uniqueness is already guaranteed by Exercise 3.6.18)

As above, we need to give a valid formulation first. By using the surjectivity of the trace theorem, see the trace theorem (Theorem 3.3.13), there exists a $\tilde{u} \in H^1(\Omega)$ such that $\tilde{u}|_{\partial\Omega} = g$ (but however such extension $\tilde{u}$ of g may not unique). From (3.6.5) we formally see that the function $w := u - \tilde{u}$ satisfies

$$(\Delta + k^2)w = f - (\Delta + k^2)\tilde{u} \text{ in } \Omega, \quad w|_{\partial\Omega} = 0. \tag{3.6.6}$$

Despite $\Delta\tilde{u}$ can be defined in distribution sense, but it may not in $H^{-1}(\Omega)$ since the integration by parts (Theorem 3.2.8) may generate some boundary terms, therefore we cannot formulate the above using similar ideas as above.

One still can perform the above simple idea by imposing additional assumptions. For example, we can further assume Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ and further assume that $g \in H^{\frac{3}{2}}(\partial\Omega)$. In this case, by the trace theorem (Theorem 3.3.14), one can construct a $\tilde{u} \in H^2(\Omega)$ such that $\tilde{u}|_{\partial\Omega} = g$. In this case $\Delta\tilde{u}$ is well-defined and is $L^2(\Omega) \subset H^{-1}(\Omega)$, therefore $f - (\Delta + k^2)\tilde{u} \in H^{-1}(\Omega)$. Hence we can solve (3.6.6) as in above. Here we do not formulate this as a theorem since the result is far away from optimal.

In order to answer Question 3.6.21, one sees that the only problematic term in (3.6.6) is the term $\Delta\tilde{u}$. One simplest way to deal with it is to find an extension $\tilde{u} \in H^1(\Omega)$ which is harmonic in Ω, i.e. $\Delta\tilde{u} = 0$ in Ω. In fact, this is true (we will not going to prove this):

THEOREM 3.6.22 (A special case of [**GT01**, Theorem 8.3], see also Theorem 4.4.6). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $g \in H^{\frac{1}{2}}(\partial\Omega)$. There exists a unique $\tilde{u} \in H^1(\Omega)$ with*

$$\Delta\tilde{u} = 0 \text{ in } \Omega, \quad \tilde{u}|_{\partial\Omega} = g.$$

Now it is not difficult to answer Question 3.6.21 in the following exercise:

EXERCISE 3.6.23. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, let $f \in H^{-1}(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $k \geq 0$ with k^2 is not an eigenvalue as mentioned in Theorem 3.6.4. Show that there exists a unique $u \in H^1(\Omega)$ satisfies (3.6.5).

CHAPTER 4

Fourier analysis, convolution and fundamental solution

As mentioned in the title of this chapter, we will introduce the Fourier series as in my previous lecture note [**Kow22**].

4.1. Fourier series

We now restrict ourselves when the case $n = 1$, and we want to compute the eigendecomposition exhibited in Theorem 3.6.4. By choosing $n = 1$ and $\Omega = (0, \pi)$ in Theorem 3.6.4, we generate an eigensystem $\{(\lambda_j, \phi_j)\}_{j\in\mathbb{N}}$ of $L^2((0,\pi))$, and $\phi_j \in C^\infty((0,\pi))$ satisfies

$$\phi_j'' = -\lambda_j \phi_j \text{ in } (0,\pi), \quad \phi(0) = \phi(\pi) = 0. \tag{4.1.1}$$

We now consider the general solution of the ODE $\phi_j'' = -\lambda_j\phi_j$ without account the boundary condition. By introducing a variable $\psi_j = \phi_j'$, we then reach

$$\begin{pmatrix} \psi_j' \\ \phi_j' \end{pmatrix} = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \begin{pmatrix} \psi_j \\ \phi_j \end{pmatrix}. \tag{4.1.2}$$

Indeed, the general solutions of (4.1.1), equivalnetly (4.1.2), forms a 2-dimentional vector space over $\mathbb{C}$, which is actually a special case of the following fundamental result:

THEOREM 4.1.1 ([**HS99**, Theorem IV-2-1]). *The general solutions of $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$, where the entries of the $n \times n$ matrix $A(t)$ are continuous on a closed interval, forms an n-dimensional vector space (over $\mathbb{C}$).*

REMARK 4.1.2. Using similar reduction (4.1.2) of (4.1.1), it is not difficult to see that the general solutions of linear ODE of order m form a m-dimensional vector space (over $\mathbb{C}$).

EXERCISE 4.1.3 ([**Bre11**, page 232]). Show that $\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$ for $j = 1, 2, \cdots$ by solving the ODE (4.1.1), and also verify that $\{\phi_j\}_{j\in\mathbb{N}}$ is orthonormal, i.e. $\|\phi_j\|_{L^2((0,\pi))} = 1$ and $(\phi_i, \phi_j)_{L^2((0,\pi))} = 0$ for all $i \neq j$.

Since $\{\phi_j\}$ is an orthonormal basis of $L^2((0,\pi))$, for each $f \in L^2((0,\pi))$, we can write

$$f(x) = \sum_{j=1}^{\infty} a_j \sin(kx) \quad \text{with} \quad a_j = \frac{2}{\pi}\int_0^\pi \sin(jx) f(x)\,\mathrm{d}x. \tag{4.1.3}$$

The expansion (4.1.3) is called the *Fourier sine series*.

EXERCISE 4.1.4. Show that the solutions of the

$$\psi_j'' = -\lambda_j \psi_j \text{ in } (0,\pi), \quad \psi'(0) = \psi'(\pi) = 0 \tag{4.1.4}$$

is $\psi_j(x) = \sqrt{\frac{2}{\pi}}\cos(jx)$ for $j = 0, 1, 2, \cdots$, and also verify that $\{\psi_j\}_{j\in\mathbb{N}}$ is orthonormal, i.e. $\|\psi_j\|_{L^2((0,\pi))} = 1$ and $(\psi_i, \psi_j)_{L^2((0,\pi))} = 0$ for all $i \neq j$.

In fact, $\{\psi_j\}_{j\in\mathbb{N}\cup\{0\}}$ also forms an orthonormal basis of $L^2((0,\pi))$. One way to to this is using the completeness of Neumann Laplacian eigenfunction (Remark 3.6.16). In this lecture note, we will later give another elementary approach (see Exercise 4.1.23 below) involving approximate identity (see Deinition 4.1.14 below). If we have the completeness, using similar ideas will induce Fourier cosine expansion for $f \in L^2((0,\pi))$:

$$f(x) = \frac{1}{2}b_0 + \sum_{j=1}^{\infty} b_j \cos(jx) \quad \text{with} \quad b_j = \frac{2}{\pi}\int_0^{\pi} \cos(jx) f(x)\,\mathrm{d}x, \tag{4.1.5}$$

see also [**Bre11**, Comments on Chapter 5]. Later (see Exercise 4.1.11 below) we will give another explaination about the normalizing constant $1/2$ in the b_0-term. Here both sine series (4.1.3) and cosine series (4.1.5) are converges in $L^2((0,\pi))$.

EXERCISE 4.1.5. Show that both sine series (4.1.3) and cosine series (4.1.5) also converges in $L^2((-\pi,\pi))$.

One should notice that sine series (4.1.3) is exactly the odd extension (2.4.10), while cosine series (4.1.5) is exactly the even extension used in Exercise 2.4.6.

EXAMPLE 4.1.6. Let $f(x) = 1$ in the interval $(0,\pi)$. The function has a Fourier sine series with coefficients

$$\begin{aligned} a_j &= \frac{2}{\pi}\int_0^{\pi} \sin(jx)\,\mathrm{d}x = -\frac{2}{\pi j}\cos(jx)\bigg|_{x=0}^{x=\pi} \\ &= \frac{2}{\pi j}(1-\cos j\pi) = \frac{2}{\pi j}(1-(-1)^j), \end{aligned}$$

which in particular gives

$$a_j = \begin{cases} \frac{4}{j\pi} & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

Thus

$$1 = \frac{4}{\pi}\sum_{j\in\mathbb{N}} \frac{1}{2j-1}\sin((2j-1)x) \quad \text{converges in } L^2((0,\pi)).$$

As mentioned in Exercise 4.1.5, the above sine series also converges in $L^2((-\pi,\pi))$. In fact, one can verify that the above series converges pointwisely (this is not easy to prove, see [**Kow22**] for details) with limit

$$f(x) = \begin{cases} 1 & , x \in (0,\pi), \\ 0 & , x = 0, \\ -1 & , x \in (-\pi,0). \end{cases}$$

EXERCISE 4.1.7. Compute the coefficients of Fourier cosine series (4.1.5) for the function $f(x) = 1$ in the interval $(0,\pi)$.

EXERCISE 4.1.8. Compute the sine series (4.1.3) and cosine series (4.1.5) for the function $f(x) = x$ in the interval $(0,\pi)$.

EXERCISE 4.1.9. Given any function $f : \mathbb{R} \to \mathbb{R}$. We say that f is odd (resp. even) if $f(-x) = -f(x)$ (resp. $f(-x) = f(x)$) for all $x \in \mathbb{R}$. Show that f can be uniquely decomposed as $f = f_{\text{even}} + f_{\text{odd}}$, where f_{even} is even and f_{odd} is odd.

In view of Exercise 4.1.9, it is natural to represent $f : (-\pi, \pi) \to \mathbb{C}$ by the following *full Fourier series*:

$$f(x) = \frac{1}{2}B_0 + \sum_{j=1}^{\infty}\left(\overbrace{A_j \sin(jx)}^{\text{odd part}} + \overbrace{B_j \cos(jx)}^{\text{even part}}\right) \quad \text{for } x \in (-\pi, \pi) \tag{4.1.6}$$

in $L^2((-\pi, \pi))$.

Exercise 4.1.10. Show that

$$\begin{aligned} A_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)\,\mathrm{d}x \quad \text{for } j = 1, 2, \cdots \\ B_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(jx)\,\mathrm{d}x \quad \text{for } j = 0, 1, 2, \cdots \end{aligned}$$

In addition, show that f is odd iff $B_j = 0$ for all $j = 0, 1, 2, \cdots$ and similarly f is even iff $A_j = 0$ for all $j = 1, 2, \cdots$.

In view of $e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$ for all $\theta \in \mathbb{R}$, see e.g. my lecture note on complex analysis [**Kow23**], it is helpful to express the Fourier series in terms of complex numbers. Alternatively, we may consider the series

$$f(x) = \sum_{j=-\infty}^{\infty} c_j e^{\mathbf{i}jx} \quad \text{for } x \in (-\pi, \pi) \quad \text{with } c_j \in \mathbb{C}. \tag{4.1.7}$$

We can write (4.1.7) as

$$\begin{aligned} f(x) &= \sum_{j=-\infty}^{\infty} c_j e^{\mathbf{i}jx} = c_0 + \sum_{j=1}^{\infty}(c_j e^{\mathbf{i}jx} + c_{-j}e^{-\mathbf{i}jx}) \\ &= c_0 + \sum_{j=1}^{\infty}(\Re c_j + \mathbf{i}\Im c_j)(\cos(jx) + \mathbf{i}\sin(jx)) \\ &\quad + \sum_{j=1}^{\infty}(\Re c_{-j} + \mathbf{i}\Im c_{-j})(\cos(jx) - \mathbf{i}\sin(jx)) \\ &= c_0 + \sum_{j=1}^{\infty}(\Re c_j \cos(jx) - \Im c_j \sin(jx)) + \mathbf{i}(\Re c_j \sin(jx) + \Im c_j \cos(jx)) \\ &\quad + \sum_{j=1}^{\infty}(\Re c_{-j}\cos(jx) + \Im c_{-j}\sin(jx)) + \mathbf{i}(-\Re c_{-j}\sin(jx) + \Im c_{-j}\cos(jx)) \\ &= c_0 + \sum_{j=1}^{\infty}(-\Im c_j + \Im c_{-j} + \mathbf{i}(\Re c_j - \Re c_{-j}))\sin(jx) \\ &\quad + \sum_{j=1}^{\infty}(\Re c_j + \Re c_{-j} + \mathbf{i}(\Im c_j + \Im c_{-j}))\cos(jx) \\ &= c_0 + \sum_{j=1}^{\infty}\mathbf{i}(c_j - c_{-j})\sin(jx) + \sum_{j=1}^{\infty}(c_j + c_{-j})\cos(jx). \end{aligned}$$

Compare this with (4.1.6), we have

$$c_0 = \frac{1}{2}B_0, \quad c_j + c_{-j} = B_j, \quad \mathbf{i}(c_j - c_{-j}) = A_j \quad \text{for all } j \in \mathbb{N},$$

equivalently,

$$c_0 = \frac{1}{2}B_0, \quad c_j = \frac{1}{2}(B_j - \mathbf{i}A_j), \quad c_{-j} = \frac{1}{2}(B_j + \mathbf{i}A_j) \quad \text{for all } j \in \mathbb{N}. \tag{4.1.8}$$

EXERCISE 4.1.11. Show that (4.1.8) is equivalent to

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{\mathbf{i}jx}\,\mathrm{d}x \quad \text{for all } j \in \mathbb{Z}.$$

Show that f is real-valued if and only if $c_j \in \mathbb{R}$ for all $j \in \mathbb{Z}$.

EXERCISE 4.1.12. Compute the Fourier series, which given in (4.1.7) and (4.1.8), of the function $f(x) = x$ in the interval $(-\pi, \pi)$.

The ideas for multi-variable case is also similar: If $f : \mathbb{R}^n \to \mathbb{C}$ is of 2π-periodic in each variable, we want to represent it by the Fourier series

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^n} c_{\boldsymbol{j}} e^{\mathbf{i}\boldsymbol{j}\cdot\boldsymbol{x}} = \sum_{\boldsymbol{j}\in\mathbb{Z}^n} c_{\boldsymbol{j}} e^{\mathbf{i}j_1x_1}\cdots e^{\mathbf{i}j_nx_n} \quad \text{for all } \boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

We now consider the cube $Q = [-\pi, \pi]^n$ and normalize the scalar product on $L^2(Q)$ by

$$(f, g) \equiv (f, g)_{L^2(Q)} := \frac{1}{|Q|}\int_Q f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} \equiv ⨍_Q f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}.$$

with $|Q| = (2\pi)^n$.

EXERCISE 4.1.13. Show that the countable set $\{e^{\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{x}}\}_{\boldsymbol{k}\in\mathbb{Z}^n}$ is an orthonormal subset of $L^2(Q)$.

In order to show that $\{e^{\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{x}}\}_{\boldsymbol{k}\in\mathbb{Z}^n}$ is a Hilbert basis of $L^2(Q)$, we need some preparations.

DEFINITION 4.1.14. A sequence $\{Q_N(\boldsymbol{x})\}_{N\in\mathbb{N}}$ of 2π-period continuous functions on the real line is called an *approximate identity* if

(1) $Q_N \geq 0$ for all $N \in \mathbb{N}$;
(2) $⨍_{-\pi}^{\pi} Q_N(x)\,\mathrm{d}x = 1$ for all $N \in \mathbb{N}$; and
(3) for each $0 < \epsilon < \pi$ one has $\lim\limits_{N\to\infty} \sup\limits_{\epsilon\leq|x|\leq\pi} Q_N(x) = 0$.

We now prove the existence of such function described in Definition 4.1.14.

LEMMA 4.1.15. *The sequence*

$$Q_N(x) := c_N\left(\frac{1+\cos x}{2}\right)^N, \quad c_N = 2\pi\left(\int_{-\pi}^{\pi}\left(\frac{1+\cos x}{2}\right)^N \mathrm{d}x\right)^{-1}$$

is an approximate identity.

PROOF. It is easy to see that $Q_N \geq 0$ and $\fint_{-\pi}^{\pi} Q_N(x)\,\mathrm{d}x = 1$ for all $N \in \mathbb{N}$. We estimate the constant c_N as followings:

$$\begin{aligned} 1 &= \frac{c_N}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1+\cos x}{2}\right)^N \mathrm{d}x = \frac{c_N}{\pi}\int_0^{\pi}\left(\frac{1+\cos x}{2}\right)^N \mathrm{d}x \\ &\geq \frac{c_N}{\pi}\int_0^{\pi}\left(\frac{1+\cos x}{2}\right)^N \sin x\,\mathrm{d}x \\ &= \frac{c_N}{\pi}\int_{-1}^{1}\left(\frac{1+t}{2}\right)^N \mathrm{d}t = \frac{2c_N}{\pi}\int_0^1 s^N\,\mathrm{d}s = \frac{2c_N}{\pi(N+1)}. \end{aligned}$$

Thus for each $0 < \epsilon < \pi$ we have

$$\begin{aligned} 0 \leq \sup_{\epsilon\leq|x|\leq\pi} Q_N(x) &\leq Q_N(\epsilon) = c_N\left(\frac{1+\cos\epsilon}{2}\right)^N \\ &\leq \frac{\pi(N+1)}{2}\left(\frac{1+\cos\epsilon}{2}\right)^N \to 0 \quad \text{as } N\to\infty \end{aligned}$$

because $0 < \frac{1+\cos\epsilon}{2} < 1$. □

Let f and g be two 2π-periodic functions. Then we formally define the convolution $f * g$ by

$$(f * g)(x) := \fint_{-\pi}^{\pi} f(y)g(x-y)\,\mathrm{d}y.$$

EXERCISE 4.1.16. Let f and g be two 2π-periodic functions. Show that $f * g = g * f$.

The following lemma explains the naming of Definition 4.1.14.

LEMMA 4.1.17. *Let Q_N be an approximate identity described in Definition 4.1.14 and let f be a 2π-periodic function. If f is continuous, then*

$$\lim_{N\to\infty} Q_N * f = f \quad \textit{converges in } L^\infty((-\pi,\pi)).$$

If $f \in L^p((-\pi,\pi))$ for some $1 \leq p < \infty$, then

$$\lim_{N\to\infty} Q_N * f = f \quad \textit{converges in } L^p((-\pi,\pi)).$$

PROOF. We first observe that

$$(Q_N * f - f)(x) = \fint_{-\pi}^{\pi} Q_N(y)(f(x-y) - f(x))\,\mathrm{d}y.$$

Case 1: f is a (uniformly) continuous 2π-periodic function. Given any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\sup_{|y|\leq\delta} |f(x-y) - f(x)| \leq \epsilon \quad \text{for all } x \in \mathbb{R}$$

and

$$\sup_{\delta\leq|x|\leq\pi} Q_N(x) \leq \epsilon \quad \text{for all sufficiently large } N.$$

Then for all sufficiently large N we estimate that

$$
\begin{aligned}
&|(Q_N * f - f)(x)| \\
&\quad \leq \frac{1}{2\pi}\left(\int_{|y|\leq\delta} + \int_{\delta\leq|y|\leq\pi}\right) Q_N(y)|f(x-y)-f(x)|\,\mathrm{d}y \\
&\quad \leq \frac{\epsilon}{2\pi}\left(\overbrace{\int_{|y|\leq\delta} Q_N(y)\,\mathrm{d}y}^{\leq 2\pi} + \overbrace{\int_{\delta\leq|y|\leq\pi} |f(x-y)-f(x)|\,\mathrm{d}y}^{\leq 4\pi\|f\|_{L^\infty(\mathbb{R})}}\right) \\
&\quad \leq \epsilon(1+2\|f\|_{L^\infty(\mathbb{R})}),
\end{aligned}
$$

which gives

$$
\limsup_{N\to\infty} |(Q_N * f - f)(x)| \leq \epsilon(1+2\|f\|_{L^\infty(\mathbb{R})}). \tag{4.1.9}
$$

By arbitrariness of $\epsilon > 0$, we conclude the first part of the lemma.

Case 2: $f \in L^p((-\pi,\pi))$ for some $1 \leq p < \infty$. Using the Minkowski's integral inequality (Exercise 1.0.10), we estimate

$$
\begin{aligned}
&\|Q_N * f - f\|_{L^p((-\pi,\pi))} \\
&\quad \leq \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} Q_N(y)(f(x-y)-f(x))\,\mathrm{d}y\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} |Q_N(y)(f(x-y)-f(x))|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \mathrm{d}y \\
&\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q_N(y)\|f(\cdot-y)-f\|_{L^p((-\pi,\pi))}\,\mathrm{d}y.
\end{aligned}
$$

Since $f \in L^p((-\pi,\pi))$, by approximate it by $C_c^\infty(-\pi,\pi)$ functions (Lemma 1.0.15), one can show that, given any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$
\sup_{|y|\leq\delta} \|f(\cdot-y)-f\|_{L^p((-\pi,\pi))} \leq \epsilon.
$$

By using (4.1.9), then for all sufficiently large N we estimate

$$
\begin{aligned}
&\|Q_N * f - f\|_{L^p((-\pi,\pi))} \\
&\quad \leq \frac{1}{2\pi}\left(\int_{|y|\leq\delta} + \int_{\delta\leq|y|\leq\pi}\right) Q_N(y)\|f(\cdot-y)-f\|_{L^p((-\pi,\pi))}\,\mathrm{d}y \\
&\quad \leq \frac{\epsilon}{2\pi}\left(\overbrace{\int_{|y|\leq\delta} Q_N(y)\,\mathrm{d}y}^{\leq 2\pi} + \overbrace{\int_{\delta\leq|y|\leq\pi} \|f(\cdot-y)-f\|_{L^p((-\pi,\pi))}\,\mathrm{d}y}^{\leq 4\pi\|f\|_{L^p((-\pi,\pi))}}\right) \\
&\quad \leq \epsilon(1+2\|f\|_{L^p((-\pi,\pi))})
\end{aligned}
$$

and we prove the second part of the lemma similar as in the first part. □

Here we also recall the following version of Hahn-Banach theorem, its contrapositive statement is quite useful in PDE:

THEOREM 4.1.18 ([**Bre11**, Corollary 1.8]). *Let X be a Banach space with dual space*[1] *X^*. Let $X_0 \subset X$ be a linear subspace. If $\overline{X_0} \neq X$, then there exists some $0 \not\equiv f \in E^*$ such that $\langle f, x \rangle_{X^* \oplus X} = 0$ for all $x \in X$.*

We are now ready to prove the following:

LEMMA 4.1.19. *$\{e^{\mathbf{i}jx}\}_{j\in\mathbb{Z}}$ is a Hilbert basis (i.e. complete orthonormal basis) of $L^2((-\pi,\pi))$.*

PROOF. Write $\phi_j(x) = e^{\mathbf{i}kx}$. Let $f \in L^2((-\pi,\pi))$ be such that

$$(f, \phi_j)_{L^2((-\pi,\pi))} = 0 \quad \text{for all } j \in \mathbb{Z}. \tag{4.1.10}$$

Let Q_N be the function described in Lemma 4.1.15, and from (4.1.10) implies $Q_N * f = 0$ for all $N \in \mathbb{N}$. Therefore, by using Lemma 4.1.17, we conclude that $f \equiv 0$, and our result immediately follows from the Hahn-Banach theorem (Theorem 4.1.18). □

EXERCISE 4.1.20. Show that $\{e^{\mathbf{i}\boldsymbol{j}\cdot\boldsymbol{x}}\}_{\boldsymbol{j}\in\mathbb{Z}^n}$ is a complete orthonormal basis of $L^2(Q)$.

We finally end this section by the following theorem, which makes the above discussions rigorous.

THEOREM 4.1.21 (Fourier series of L^2 functions). *If $f \in L^2(Q)$, then one has the Fourier series*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \hat{f}(\boldsymbol{k}) e^{\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{x}} \quad \textit{converges in } L^2(Q) \tag{4.1.11}$$

with the Fourier coefficients

$$\hat{f}(\boldsymbol{k}) = \fint_Q f(\boldsymbol{x}) e^{-\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x}. \tag{4.1.12}$$

One has the Parseval identity

$$\|f\|_{L^2(Q)}^2 = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\hat{f}(\boldsymbol{k})|^2.$$

Conversely, if the sequence $\{c_{\boldsymbol{k}}\} \in \ell^2(\mathbb{Z}^n)$, i.e. $\sum_{\boldsymbol{k}\in\mathbb{Z}^n} |c_{\boldsymbol{k}}|^2 < \infty$, then the series $\sum_{\boldsymbol{k}\in\mathbb{Z}^n} c_{\boldsymbol{k}} e^{\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ converges in $L^2(Q)$ to some $f \in L^2(Q)$ and it is necessarily $c_{\boldsymbol{k}} = \hat{f}(\boldsymbol{k})$, i.e. the Fourier series is unique.

REMARK 4.1.22. If we equipped $\ell^2(\mathbb{Z}^n)$ with the norm

$$\|\{c_{\boldsymbol{k}}\}\|_{\ell^2(\mathbb{Z}^n)} := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^n} |c_{\boldsymbol{k}}|^2 \right)^{\frac{1}{2}},$$

then in fact it is a Banach space. Theorem 4.1.21 says that there is a 1-1 corresponding between the elements in $L^2(Q)$ with the elements in $\ell^2(\mathbb{Z}^n)$. Therefore the Fourier coefficients (4.1.12) can be viewed as the discrete Fourier transform, and the inverse discrete Fourier transform is given by the formula (4.1.11). We will further explain this in next section.

[1] I didn't explain the meaning of the "dual space of Banach space", but let's keep the terminology here for future references.

PROOF OF THEOREM 4.1.21. The first part is an immediate consequence of Exercise 4.1.20. We now prove the uniqueness of the Fourier series. If $\{c_{\boldsymbol{k}}\} \in \ell^2(\mathbb{Z}^n)$, then we see that the partial sum

$$S_N(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|\le N} c_{\boldsymbol{k}} e^{\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$

is a Cauchy sequence in $L^2(Q)$. Since $L^2(Q)$ is Banach, then we know that $S_N(\boldsymbol{x})$ converges to some $f \in L^2(Q)$ as $N \to \infty$. For each $N \ge k$, we also see that

$$|c_{\boldsymbol{k}} - \hat{f}(\boldsymbol{k})| = |(S_N - f, e^{\mathbf{i}\langle \boldsymbol{k},\cdot\rangle})| \le C_n \|e^{\mathbf{i}\langle \boldsymbol{k},\cdot\rangle}\|_{L^2(Q)} \|S_N - f\|_{L^2(Q)} \to 0 \quad \text{as } N \to \infty,$$

which conclude $c_{\boldsymbol{k}} = \hat{f}(\boldsymbol{k})$. □

EXERCISE 4.1.23. Show that $\{\psi_j\}_{j\in\mathbb{N}\cup\{0\}}$ given in Exercise 4.1.4 also forms an orthonormal basis of $L^2((0,\pi))$, therefore the Neumann series discussed above is valid. [Hint: Use Exercise 4.1.10.]

It is possible to discuss the convergence of Fourier series in different sense (e.g. pointwise convergence, absolute convergence as well as uniform convergence), one can refer to my lecture note [**Kow22**]. We will not discuss them in this lecture note.

4.2. A quick introduction of Fourier transform

As mentioned in Remark 4.1.22, the Fourier series is a discrete version of Fourier transform. We now start to explain this from the following exercise:

EXERCISE 4.2.1. Let $T > 0$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a function with period $2T$ on each variable. Show that the Fourier series of f is given by

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \hat{f}(\boldsymbol{k}) e^{i\frac{\pi}{T}\boldsymbol{k}\cdot\boldsymbol{x}} \quad \text{with} \quad \hat{f}(\boldsymbol{k}) = ⨍_{[-T,T]^n} f(\boldsymbol{y}) e^{-\mathbf{i}\frac{\pi}{T}\boldsymbol{k}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}, \tag{4.2.1}$$

where $⨍_{[-T,T]^n}$ is the average integral given by

$$⨍_{[-T,T]^n} \equiv \frac{1}{|[-T,T]^n|}\int_{[-T,T]^n} = \frac{1}{(2T)^n}\int_{[-T,T]^n}.$$

If we denote $\boldsymbol{\xi} = \frac{\pi}{T}\boldsymbol{k} \in \frac{\pi}{T}\mathbb{Z}^n$, then the scaled Fourier series (4.2.1) can be rewritten as

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\sum_{\boldsymbol{k}\in\mathbb{Z}^n}\left(\int_{[-T,T]^n} f(\boldsymbol{y})e^{-\mathbf{i}\boldsymbol{\xi}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\right) e^{\mathbf{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(\frac{\pi}{T}\right)^n.$$

We observe that $\left(\frac{\pi}{T}\right)^n$ is the volume of each square in the mesh $\frac{\pi}{T}\mathbb{Z}^n$. In view of Riemann integral, formally taking the limit $T \to \infty$ we see that

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(\boldsymbol{y})e^{-\mathbf{i}\boldsymbol{\xi}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\right) e^{\mathbf{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{\xi}. \tag{4.2.2}$$

DEFINITION 4.2.2. The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is defined by

$$(\mathscr{F}f)(\boldsymbol{\xi}) \equiv \hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^n} f(\boldsymbol{y})e^{-\mathbf{i}\boldsymbol{\xi}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}.$$

From this, it is easy to see that $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$ and $\hat{f}$ is continuous.

From (4.2.2), we formally have the Fourier inversion formula

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) e^{\mathbf{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} \, \mathrm{d}\boldsymbol{\xi} \equiv \frac{1}{(2\pi)^n} (\mathscr{F} f)(-\boldsymbol{x}). \tag{4.2.3}$$

But the problem here is the regularity does not match since we do not know whether the Fourier transform converges for $L^\infty(\mathbb{R}^n)$ functions or not, because we only define Fourier transform for $L^1(\mathbb{R}^n)$-functions. Our goal to explain when does the equality (4.2.3) is well-defined.

Exercise 4.2.3. Using Fubini's theorem (Theorem 1.0.4), show that

$$\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) g(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \hat{g}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \quad \text{for all } f, g \in L^1(\mathbb{R}^n).$$

The above exercise might suggests defining the Fourier transform $\hat{f}$ of a distribution $f \in \mathscr{D}'(\mathbb{R}^n)$. However, this idea does not work since we do not whether $\mathscr{F}(C_c^\infty(\mathbb{R}^n))$ is contained in $C_c^\infty(\mathbb{R}^n)$. In fact, the failure of this idea is confirmed by the following fact:

Lemma 4.2.4. *If $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\hat{\varphi} \in C_c^\infty(\mathbb{R}^n)$, then $\varphi \equiv 0$.*

This fact says that $\int_{\mathbb{R}^n} f(\boldsymbol{x}) \hat{g}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$ is not well-defined for general distribution $f \in \mathscr{D}'(\mathbb{R}^n)$. One can overcome this difficulty by introduce the following class of functions:

Definition 4.2.5. The Schwartz class of rapidly decreasing function is defined as

$$\mathscr{S}(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(\mathbb{R}^n) : \sum_{|\boldsymbol{\alpha}| \le m} \sup_{x \in \mathbb{R}^n} (1 + |\boldsymbol{x}|^2)^{\frac{m}{2}} |\partial^{\boldsymbol{\alpha}} \varphi(\boldsymbol{x})| < \infty \text{ for all } m \in \mathbb{Z}_{\ge 0} \right\}.$$

Exercise 4.2.6. For each $1 \le p < \infty$, show that $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$.

Similar to distributions, it is also possible to define suitable topology for $\mathscr{S}(\mathbb{R}^n)$. In fact, the Fourier inversion formula (4.2.3) can be done in a rigorous way:

Theorem 4.2.7 (Fourier inversion formula). *The Fourier transform is an algebraic and topological isomorphism, i.e. $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a well-defined bijective mapping, which is continuous, and its inverse also continuous. In addition, its inverse is the operator $\mathscr{F}^{-1} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is given by the formula*

$$(\mathscr{F}^{-1} g)(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} g(\boldsymbol{\xi}) e^{\mathbf{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} \, \mathrm{d}\boldsymbol{\xi} \tag{4.2.4}$$

for all $g \in \mathscr{S}(\mathbb{R}^n)$ and for all $\boldsymbol{x} \in \mathbb{R}^n$.

Exercise 4.2.8. For each $f, g \in \mathscr{S}(\mathbb{R}^n)$, show that

1. **Symmetry.** $\mathscr{F}^2 f = (2\pi)^n \tilde{f}$ with $\tilde{f}(\boldsymbol{x}) = f(-\boldsymbol{x})$. Consequently, $\mathscr{F}^4 f = (2\pi)^{2n} f$.
2. **Parseval's identity.** $\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{x}) g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \hat{g}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$
3. **Parseval's identity.** $\int_{\mathbb{R}^n} f(\boldsymbol{x}) \overline{g(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) \overline{\hat{g}(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi}$. Consequently, $\|f\|_{L^2(\mathbb{R}^n)}^2 = (2\pi)^{-n} \|\hat{f}\|_{L^2(\mathbb{R}^n)}^2$.
4. **Derivative.** $\mathscr{F}(\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f)(\boldsymbol{x}) = (\mathbf{i}\boldsymbol{\xi})^{\boldsymbol{\alpha}} \hat{f}(\boldsymbol{\xi})$, where $\boldsymbol{y}^{\boldsymbol{\alpha}} := y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n}$.
5. **Polynomial.** $\mathscr{F}((-\mathbf{i}\boldsymbol{x})^{\boldsymbol{\beta}} f)(\boldsymbol{\xi}) = \partial_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} \hat{f}(\boldsymbol{\xi})$.

Suppose that A is a real symmetric positive definite matrix (in the sense of Exercise 2.3.1). Compute $\mathscr{F}(-\Delta f)$, where Δ is the Laplacian (see again Remark 2.3.5).

Definition 4.2.9. We denote the *tempered distribution* $\mathscr{S}'(\mathbb{R}^n)$ the collection of bounded linear functional on $\mathscr{S}(\mathbb{R}^n)$.

By using the Fourier inversion formula on Schwartz function (Theorem 4.2.7), suggested by Exercise 4.2.3, one can define the Fourier transform $\hat{f}$ of $f \in \mathscr{S}'(\mathbb{R}^n)$ by

$$\langle \hat{f}, g\rangle_{\mathscr{S}'(\mathbb{R}^n)\oplus\mathscr{S}(\mathbb{R}^n)} := \langle f, \hat{g}\rangle_{\mathscr{S}'(\mathbb{R}^n)\oplus\mathscr{S}(\mathbb{R}^n)} \quad \text{for all } g \in \mathscr{S}(\mathbb{R}^n). \tag{4.2.5}$$

In fact, the Fourier inversion formula also holds true for tempered distributions:

Theorem 4.2.10 (Fourier inversion formula). *The Fourier transform is a bijective map from $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, and the Fourier inversion formula (4.2.4) also holds.*

Remark 4.2.11. All properties in Exercise 4.2.8 also can be extended for the Fourier transform in Theorem 4.2.10 as well. Even though the precise definition is given by (4.2.5), we usually still denote as in Definition 4.2.2.

4.3. Distribution with compact support and convolution

We now want to extend the convolution for functions (Definition 1.0.12) as in [**FJ98**], but here we only exhibit some special cases which we needed. In order to do so, we first introduce some concept of tensor products. Given any functions f, g, the tensor product $f \otimes g$ is defined by

$$(f \otimes g)(\boldsymbol{x}, \boldsymbol{y}) := f(\boldsymbol{x})g(\boldsymbol{y}).$$

For each $f \in L^1_{\rm loc}(\mathbb{R}^n)$, let $T_f \in \mathscr{D}'(\mathbb{R}^n)$ be the distribution given in Example 3.1.17, and this suggests us to define

$$(T_f \otimes T_g)(\varphi) := \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} f(\boldsymbol{x})g(\boldsymbol{y})\varphi(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^m)$. If one take a particular choice $\varphi(\boldsymbol{x}, \boldsymbol{y}) = (\varphi_1 \otimes \varphi_2)(\boldsymbol{x}, \boldsymbol{y}) = \varphi_1(\boldsymbol{x})\varphi_2(\boldsymbol{y})$, one immediately obtain

$$(T_f \otimes T_g)(\varphi_1 \otimes \varphi_2) = T_f(\varphi_1)T_g(\varphi_2).$$

However, not all φ can be written in the form of $\varphi_1 \otimes \varphi_2$, and not all distributions can be written in the form of T_f, therefore the well-definedness of the tensor product is not so obvious. Despite it is not so obvious, however it can be done:

Theorem 4.3.1. [**FJ98**, Theorem 4.3.2] *Given any $T \in \mathscr{D}'(\mathbb{R}^n)$ and $S \in \mathscr{D}'(\mathbb{R}^m)$, there exists a unique element $T \otimes S \in \mathscr{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, which is called the* tensor product *of T and S, such that*

$$(T \otimes S)(\varphi_1 \otimes \varphi_2) = T(\varphi_1)S(\varphi_2) \quad \textit{for all } \varphi_1 \in C_c^\infty(\mathbb{R}^n) \textit{ and } \varphi_2 \in C_c^\infty(\mathbb{R}^m).$$

Here we refer to [**FJ98**, Theorem 4.3.3] for some basic properties of tensor products. In order to motivate the definition of convolution, let us again consider the distribution $T_f \in \mathscr{D}'(\mathbb{R}^n)$ be the distribution given in Example 3.1.17 with $f \in L^1_{\rm loc}(\mathbb{R}^n)$. By direct computations, one sees that

$$\begin{aligned}(T_f * T_g)(\varphi) &= \int_{\mathbb{R}^n} (f*g)(\boldsymbol{z})\varphi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(\boldsymbol{y})f(\boldsymbol{z}-\boldsymbol{y})\varphi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\boldsymbol{x})g(\boldsymbol{y})\varphi(\boldsymbol{x}+\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\boldsymbol{x})g(\boldsymbol{y})\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \quad \text{where } \phi(\boldsymbol{x},\boldsymbol{y}) = \varphi(\boldsymbol{x}+\boldsymbol{y}) \\ &= (T_f \otimes T_g)(\phi).\end{aligned}$$

This suggests us to define the convolution of distirbutions $T, S \in \mathscr{D}'(\mathbb{R}^n)$ by

$$(T * S)(\varphi) := (T \otimes S)(\phi) \quad \text{with} \quad \phi(\boldsymbol{x},\boldsymbol{y}) = \varphi(\boldsymbol{x}+\boldsymbol{y}). \tag{4.3.1}$$

Formally from (4.3.1) we immediately sees that

$$T * S = S * T.$$

However, one cannot guarantee ϕ has compact support in $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ even though with $\varphi \in C_c^\infty(\mathbb{R}^n)$. In order to encounter this problem, we introduce the following notion:

Definition 4.3.2. For any open set $\Omega \subset \mathbb{R}^n$, the tempered distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is said to *vanish* on Ω, we often denoted as $T = 0$ in Ω, if

$$T(\varphi) = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Two tempered distributions $T_1, T_2 \in \mathscr{D}'(\mathbb{R}^n)$ are said to be equal in Ω if $T_1 - T_2$ vanish in Ω. The *support* of a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$, denoted by $\mathrm{supp}\,(T)$, is the complement of the largest open subset of $\mathbb{R}^n$ where T vanishes (therefore $\mathrm{supp}\,(T)$ is necessarily closed in $\mathbb{R}^n$). Accordingly, we say that a tempered distribution has *compact support* if its support is a compact set in $\mathbb{R}^n$.

One can define a suitable topology (not similar to above) for $\mathscr{E}(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n)$, and we let $\mathscr{E}'(\mathbb{R}^n)$ be the set of continuous linear functional on $\mathscr{E}(\mathbb{R}^n)$. Sometimes we denote $\mathscr{E}(\mathbb{R}^n)$ to emphasize the topology, but however we will still denote $C^\infty(\mathbb{R}^n)$ in this lecture note. In fact, the above notions are actually consistent:

Theorem 4.3.3. *Let $T \in \mathscr{D}'(\mathbb{R}^n)$. The following are equivalent:*

(1) *T has compact support;*
(2) *T can be extended to an element in $\mathscr{E}'(\mathbb{R}^n)$.*

From this, if a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ has compact support, then $T(\varphi)$ is well-defined for general smooth function $\varphi \in C^\infty(\mathbb{R}^n)$ without compact support, this fact is helpful to deal with the problem encountered while define the convolution of distributions (4.3.1). This strongly suggests us to overcome this difficulty is to assume that one of the distributions T or S has compact support.

Theorem 4.3.4. *The convolution of distributions* (4.3.1) *is a well-defined* (*separately continuous*[2]) *map*

[2] We will not explain this topological terminology in this lecture note, but we still keep here for future reference.

(1) $\mathscr{E}'(\mathbb{R}^n) \times \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$;
(2) $\mathscr{E}'(\mathbb{R}^n) \times \mathscr{E}'(\mathbb{R}^n) \to \mathscr{E}'(\mathbb{R}^n)$;

One can refer [**FJ98**] how to deal with the convolution of distributions with non-compact supports. Despite the precise definition of convolution is given by (4.3.1), but here and after we abuse the notation by using the notation same as the convolution for functions (Definition 1.0.12). Similar to Lemma 3.2.10 and Lemma 3.2.11, it is worthmentioning that similar properties also holds for distirbutional derivatives:

LEMMA 4.3.5. *Let $T \in \mathscr{E}'(\mathbb{R}^n)$ and $S \in \mathscr{D}'(\mathbb{R}^n)$, then*

$$\partial^{\boldsymbol{\alpha}}(T * S) = (\partial^{\boldsymbol{\alpha}} T) * S = T * (\partial^{\boldsymbol{\alpha}} S).$$

However, one should be careful about the associativity. The proper statement should be the followings:

LEMMA 4.3.6. *Let $T \in \mathscr{E}'(\mathbb{R}^n)$, $S \in \mathscr{D}'(\mathbb{R}^n)$ and $R \in \mathscr{E}'(\mathbb{R}^n)$, then*

$$R * (S * T) = (R * S) * T = (R * T) * S.$$

However, the above associativity property may fails without assuming the compact support condition properly, we provide a counterexample in the following exercise:

EXERCISE 4.3.7. Let $T_1 = 1$, $T_2 = \delta_0'$ (derivative of Dirac distribution at 0) and $T_3 = H$ (Heaviside function given in (3.1.1)). Note that both T_1 and T_3 do not have compact support on $\mathbb{R}^1$. Show that

$$(T_1 * T_2) * T_3 \text{ and } T_1 * (T_2 * T_3) \text{ both exist but they are not identical.}$$

4.4. Fundamental solution of Laplacian

For simplicity, here we only consider the Laplacian. The ideas in this section can be generalized for general elliptic systems, here we refer to the monograph [**McL00**]. We are now interested to construct a distribution $\Phi \in \mathscr{S}'(\mathbb{R}^n)$ for which

$$-\Delta \Phi = \delta_0 \quad \text{in } \mathscr{S}'(\mathbb{R}^n). \tag{4.4.1}$$

In view of Lemma 4.3.5, for each $f \in C_c^\infty(\mathbb{R}^n)$, one can easily see that

$$-\Delta(\Phi * f) = -\Delta\Phi * f = \delta_0 * f = f,$$

which immediately gives a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$, which motivate us to study such distribution Φ. Taking the Fourier transform on (4.4.1) suggests us to find $\hat{\Phi}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-2}$. However this seems not a good idea since there is a singularity at $\boldsymbol{\xi} = \mathbf{0}$. For simplicity, here we will only consider Laplacian case, one can refer e.g. [**DHM18**] for discussions for more general elliptic systems.

We now define

$$\Phi(\boldsymbol{x}) := \begin{cases} \frac{1}{n(n-2)|B_1|}|\boldsymbol{x}|^{2-n} & \text{when } n > 2, \\ -\frac{1}{2\pi}\log|\boldsymbol{x}| & \text{when } n = 2, \end{cases} \tag{4.4.2}$$

where $|B_1|$ is the volume of unit ball in $\mathbb{R}^n$, and in fact it is given by

$$|B_1| = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

EXERCISE 4.4.1. Show that $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and it satisfies (4.4.1), more precisely, show that

$$-\int_{\mathbb{R}^n} \Phi(\boldsymbol{x})\Delta u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = u(0) \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n).$$

[Hint: Choose $R > 0$ such that $\operatorname{supp}(f) \subset B_R$. Given $\epsilon > 0$, consider the integral $\int_{\mathbb{R}^n \setminus B_\epsilon} \Phi(\boldsymbol{x})\Delta f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{B_R \setminus B_\epsilon} \Phi(\boldsymbol{x})\Delta f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, and then doing integration by parts on $B_R \setminus B_\epsilon$. Be careful the orientation on the inner sphere ∂B_ϵ.]

EXERCISE 4.4.2. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Prove the Green's representation formula:

$$u(\boldsymbol{y}) = \int_{\partial\Omega} \left(\Phi(\boldsymbol{x}-\boldsymbol{y})\partial_{\boldsymbol{\nu}} u(\boldsymbol{x}) - u(\boldsymbol{x})\partial_{\boldsymbol{\nu}}\Phi(\boldsymbol{x}-\boldsymbol{y})\right)\mathrm{d}S_{\boldsymbol{x}} - \int_\Omega \Phi(\boldsymbol{x}-\boldsymbol{y})\Delta u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

[Hint: Can be done using similar ideas as in previous exercise.]

DEFINITION 4.4.3. The integral $\int_\Omega \Phi(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ is call the Newtonian potential with density f.

If we choose u be the unique solution of $\Delta u = 0$ in Ω with $u|_{\partial\Omega} = g$ (see Theorem 3.6.22), from Exercise 4.4.2 we have

$$u(\boldsymbol{y}) = \int_{\partial\Omega} \left(\Phi(\boldsymbol{x}-\boldsymbol{y})\partial_{\boldsymbol{\nu}} u(\boldsymbol{x}) - g(\boldsymbol{x})\partial_{\boldsymbol{\nu}}\Phi(\boldsymbol{x}-\boldsymbol{y})\right)\mathrm{d}S_{\boldsymbol{x}}. \tag{4.4.3}$$

Now suppose that $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $\Delta h = 0$ in Ω, then by integration by parts one can easily show that

$$0 = \int_\Omega h(\boldsymbol{x})\Delta u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} \left(h(\boldsymbol{x})\partial_{\boldsymbol{\nu}} u(\boldsymbol{x}) - g(\boldsymbol{x})\partial_{\boldsymbol{\nu}} h(\boldsymbol{x})\right)\mathrm{d}S_{\boldsymbol{x}}. \tag{4.4.4}$$

In view of (4.4.3) and (4.4.4), if we write $G(\boldsymbol{x},\boldsymbol{y}) := \Phi(\boldsymbol{x}-\boldsymbol{y}) - h(\boldsymbol{x})$, then we reach

$$u(\boldsymbol{y}) = \int_{\partial\Omega} \left(G(\boldsymbol{x},\boldsymbol{y})\partial_{\boldsymbol{\nu}} u(\boldsymbol{x}) - g(\boldsymbol{x})\partial_{\boldsymbol{\nu}} G(\boldsymbol{x},\boldsymbol{y})\right)\mathrm{d}S_{\boldsymbol{x}}.$$

If we can find h such that $G(\boldsymbol{x},\boldsymbol{y}) = 0$ for all $\boldsymbol{x} \in \partial\Omega$, then we reach

$$u(\boldsymbol{y}) = -\int_{\partial\Omega} g(\boldsymbol{x})\partial_{\boldsymbol{\nu}} G(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{x}}, \tag{4.4.5}$$

which gives a representation for the unique solution u of $\Delta u = 0$ in Ω with $u|_{\partial\Omega} = g$ (see Theorem 3.6.22). Such function G in (4.4.5) is called the *Green's function*.

When Ω is a ball, indeed the Green function can be written explicitly, and thus (4.4.5) is valid at least when Ω is a ball:

EXERCISE 4.4.4 (Poisson integral formula). Let $\Omega = B_R = B_R(0) = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| < R\}$. For each $\mathbf{0} \neq \boldsymbol{x} \in \mathbb{R}^n$, let $\tilde{\boldsymbol{x}}$ be its reflection with respect to ∂B_R with formula

$$\tilde{\boldsymbol{x}} := \frac{R^2}{|\boldsymbol{x}|^2}\boldsymbol{x} \quad \text{for all } \mathbf{0} \neq \boldsymbol{x} \in \mathbb{R}^n.$$

Let Φ be the function given in (4.4.2), and define

$$G(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \Phi(\boldsymbol{x}-\boldsymbol{y}) - \Phi\left(\dfrac{|\boldsymbol{y}|}{R}(\boldsymbol{x}-\tilde{\boldsymbol{y}})\right) & , \boldsymbol{y} \neq \mathbf{0}, \\ \Phi(\boldsymbol{x}) - \Phi(R) & , \boldsymbol{y} = \mathbf{0} \end{cases}$$

(1) Show that $G(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y},\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x},\boldsymbol{y} \in \overline{B_R}$.
(2) Show that

$$\partial_{\boldsymbol{\nu}} G(\boldsymbol{x},\boldsymbol{y}) \equiv \partial_{|\boldsymbol{x}|} G(\boldsymbol{x},\boldsymbol{y}) = \frac{R^2 - |\boldsymbol{y}|^2}{n|B_1|R} |\boldsymbol{x}-\boldsymbol{y}|^{-n} \geq 0 \tag{4.4.6}$$

for all $\boldsymbol{x} \in \partial B_R$ and $\boldsymbol{y} \in B_R$.
(3) If $u \in C^2(B_R) \cap C^1(\overline{B_R})$ satisfies $\Delta u = 0$ in Ω, then

$$u(\boldsymbol{y}) = \frac{R^2 - |\boldsymbol{y}|^2}{n|B_1|R} \int_{\partial B_R} \frac{u(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|^n} \,\mathrm{d}S_{\boldsymbol{x}} \quad \text{for all } \boldsymbol{y} \in B_R. \tag{4.4.7}$$

We usually call (4.4.6) the *Poisson kernel*, and we denoted it by

$$K(\boldsymbol{x},\boldsymbol{y}) := \frac{R^2 - |\boldsymbol{y}|^2}{n|B_1|R} |\boldsymbol{x}-\boldsymbol{y}|^{-n} \quad \text{for all } \boldsymbol{x} \in \partial B_R \text{ and } \boldsymbol{y} \in B_R.$$

By choosing $u \equiv 1$ in (4.4.7), one immediately sees that

$$\int_{\partial B_R} K(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}S_{\boldsymbol{x}} = 1 \quad \text{for all } \boldsymbol{y} \in B_R. \tag{4.4.8}$$

Based on this observation, we now able to solve the Dirichlet problem in the following sense (it is interesting to compare this result with Theorem 3.6.22):

THEOREM 4.4.5. *Let $\Omega = B_R = B_R(0) = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| < R\}$ and let φ be a continuous function on ∂B_R. Then the function u defined by*

$$u(\boldsymbol{x}) = \begin{cases} \int_{\partial B_R} K(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y}) \,\mathrm{d}S_{\boldsymbol{y}} & \text{for } \boldsymbol{x} \in B_R, \\ g(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \partial B_R, \end{cases}$$

belongs to $C^2(B_R) \cap C^0(\overline{B_R})$ and satisfies $\Delta u = 0$ in Ω with $u|_{\partial\Omega} = g$.

PROOF. It remains to show that $u \in C^0(\overline{B_R})$. From (4.4.8) it is easy to see that

$$u(\boldsymbol{x}) - u(\boldsymbol{x}_0) = \int_{\partial B_R} K(\boldsymbol{x},\boldsymbol{y})(g(\boldsymbol{y}) - g(\boldsymbol{x}_0)) \,\mathrm{d}S_{\boldsymbol{y}}.$$

Given any $\boldsymbol{x}_0 \in \partial B_R$ and let $\epsilon > 0$ be arbitrary number. Since φ is (uniformly) continuous, one can find $\delta = \delta(\epsilon) > 0$ such that

$$\sup_{|\boldsymbol{x}-\boldsymbol{x}_0|<\delta} |\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}_0)| \leq \epsilon.$$

Then if $|\boldsymbol{x}-\boldsymbol{x}_0| < \delta/2$, then from (4.4.6) we see that

$$\begin{aligned} |u(\boldsymbol{x}) - u(\boldsymbol{x}_0)| &= \left| \int_{\partial B_R} K(\boldsymbol{x},\boldsymbol{y})(g(\boldsymbol{y}) - g(\boldsymbol{x}_0)) \,\mathrm{d}S_{\boldsymbol{y}} \right| \\ &\leq \int_{\boldsymbol{y} \in \partial B_R, |\boldsymbol{y}-\boldsymbol{x}_0| \leq \delta} K(\boldsymbol{x},\boldsymbol{y}) |g(\boldsymbol{y}) - g(\boldsymbol{x}_0)| \,\mathrm{d}S_{\boldsymbol{y}} \\ &\quad + \int_{\boldsymbol{y} \in \partial B_R, |\boldsymbol{y}-\boldsymbol{x}_0| > \delta} K(\boldsymbol{x},\boldsymbol{y}) |g(\boldsymbol{y}) - g(\boldsymbol{x}_0)| \,\mathrm{d}S_{\boldsymbol{y}} \\ &\leq \epsilon + \frac{2\|g\|_{L^\infty(\partial B_R)}(R - |\boldsymbol{x}|^2) R^{n-2}}{(\delta/2)^n}. \end{aligned}$$

Since $\boldsymbol{x}_0 \in \partial B_R$, then $\boldsymbol{x} \to \boldsymbol{x}_0$ implies $|\boldsymbol{x}| \to R$, therefore

$$\limsup_{\boldsymbol{x}\to\boldsymbol{x}_0} |u(\boldsymbol{x}) - u(\boldsymbol{x}_0)| \le \epsilon.$$

By arbitrariness of $\epsilon > 0$, we conclude our theorem. □

In fact, one can show the unique solvable of the Dirichlet problem $\Delta u = 0$ in Ω with $u|_{\partial\Omega} = g$ for some suitable regular $\partial\Omega$ by using the method of subharmonic functions, called the Perron's method. This ideas even works for elliptic equations, see e.g. [**GT01**, Theorem 6.13]. In fact, we have the following result, which is even holds true for general elliptic equations:

THEOREM 4.4.6 (A special case of [**GT01**, Theorem 6.13], see also Theorem 3.6.22). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $g \in C^0(\partial\Omega)$. There exists a unique $\tilde{u} \in H^1_{\rm loc}(\Omega) \cap C^0(\overline{\Omega})$ with*

$$\Delta u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = g.$$

Here $H^1_{\rm loc}(\Omega) = \{u \in \mathscr{D}'(\Omega) : u|_\omega \in H^1(\omega)$ for any open set ω satisfies $\overline{\omega} \subset \Omega\}$.

Fix any $\boldsymbol{x} \in \Omega$. By using Theorem 4.4.6, one sees that the mapping $g \mapsto u(\boldsymbol{x})$ is a linear functional on $C^0(\partial\Omega)$, which is positive in the sense of $g \ge 0$ implies $u(\boldsymbol{x}) \ge 0$ in Ω and if $0 \not\equiv g \ge 0$ implies $u(\boldsymbol{x}) > 0$, which holds true by strong maximum principle (Lemma 3.5.6). Therefore, by the Riesz representation theorem [**Rud87**, Theorem 6.19], there exists a Borel measure $\mu_{\boldsymbol{x}}$ on $\partial\Omega$ such that

$$u(\boldsymbol{x}) = \int_{\partial\Omega} g(\boldsymbol{y}) \,\mathrm{d}\mu_{\boldsymbol{x}}(\boldsymbol{y}).$$

This is related to the existence of the Green's function in (4.4.5) by interpreting

$$-\partial_{\boldsymbol{\nu}} G(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}S_{\boldsymbol{x}} := \mathrm{d}\mu_{\boldsymbol{x}}(\boldsymbol{y}).$$

DEFINITION 4.4.7. We call such measures $\{\mu_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\Omega}$ the *harmonic measures*.

Here we refer the monographs [**CS05, Ken95**] for further introduction on this topic, see also [**CFMS81**] for discussions about some generalizations.

CHAPTER 5

Partial differential equation in weak sense (continued)

In Chapter 2, we discussed wave equation under classical sense. Recall that in the very beginning of Chapter 3, we give some reason that it is necessarily to investigate the PDE in weak sense. We now turn back to study the wave equation, but now in weak sense, as in [**Eva10**, Chapter 7]. For simplicity, here we only consider constant coefficient case. The argument can be easily extended for variable case. The ideas in this chapter even works for some PDE involving pseudodifferential operators with some necessarily modifications [**KLW22, KMS23, KW23**].

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and for each $T > 0$ we denote $\Omega_T := (0,T)\times\Omega$ and $(\partial\Omega)_T := (0,T)\times\partial\Omega$. Our goal is to solve the following *hyperbolic initial-boundary value problem* for some suitable external source $f = f(t,\boldsymbol{x})$ as well as initial conditions $g = g(\boldsymbol{x})$ and $h = h(\boldsymbol{x})$:

$$(5.0.1)\qquad \begin{cases} \partial_t^2 u - \Delta u + \boldsymbol{b}\cdot\nabla u + cu = f & \text{in } \Omega_T, \\ u = 0 & \text{on } (\partial\Omega)_T, \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = h(\boldsymbol{x}) & \text{for all } \boldsymbol{x}\in\Omega. \end{cases}$$

Here $\boldsymbol{b} = (b_1,\cdots,b_n)$ and c are constants (for simplicity).

5.1. Formulation of weak solutions

We first explain the motivation for definition of weak solution. Fix any $t \in (0,T)$ and given any $\varphi = \varphi(\boldsymbol{x}) \in C_c^\infty(\Omega)$, from (5.0.1) and integration by parts (Theorem 3.2.8) formally one has

$$\langle f(t,\cdot),\varphi\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)} = \partial_t^2 (u(t,\cdot),\varphi)_{L^2(\Omega)} + a(u(t,\cdot),\varphi),$$

where the bilinear form B is given by

$$a(\phi,\varphi) := (\nabla\phi,\nabla\varphi)_{L^2(\Omega)} + (\boldsymbol{b}\cdot\nabla\phi,\varphi)_{L^2(\Omega)} + c(\phi,\varphi)$$

This strongly suggests us to work with $u(t,\cdot) \in H_0^1(\Omega)$ and $f(t,\cdot) \in H^{-1}(\Omega)$. For later convenience, we define the associated mapping

$$\tilde{u} : [0,T] \to H_0^1(\Omega), \quad [\tilde{u}(t)](\boldsymbol{x}) := u(t,\boldsymbol{x}),$$
$$\tilde{f} : [0,T] \to H^{-1}(\Omega), \quad [\tilde{f}(t)](\boldsymbol{x}) := f(t,\boldsymbol{x}).$$

We denote

$$(\tilde{u}'(t))(\boldsymbol{x}) = \partial_t u(t,\boldsymbol{x}), \quad (\tilde{u}''(t))(\boldsymbol{x}) = \partial_t^2 u(t,\boldsymbol{x}),$$

and we write

$$\langle \tilde{f}(t),\varphi\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)} = \partial_t^2 (\tilde{u}(t),\varphi)_{L^2(\Omega)} + B(\tilde{u},v;t),$$

with

$$B[\tilde{u},\varphi;t] := a(u(t,\cdot),\varphi).$$

For convenience, for each Hilbert space H, we denote

$$L^2(0,T;H) := \left\{ u : (0,T) \times H \to \mathbb{R} : t \mapsto \|u(t,\cdot)\|_H \in L^2((0,T)) \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{L^2(0,T;H)} := \left(\int_0^T \|u(t,\cdot)\|_H^2 \, \mathrm{d}t \right)^{\frac{1}{2}}. \tag{5.1.1}$$

EXERCISE 5.1.1. Verify that (5.1.1) is a norm.

We now state the precise meaning of the weak solution of (5.0.1):

DEFINITION 5.1.2. We say a function $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;L^2(\Omega))$ and $\partial_t^2 \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of the hyperbolic IBVP (5.0.1) if

$$\langle \tilde{u}'', \varphi \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)} + B[\tilde{u}, \varphi; t] = \langle \tilde{f}, \varphi \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)} \tag{5.1.2}$$

for all $\varphi \in H_0^1(\Omega)$ and a.e. $t \in (0,T)$, and $\tilde{u}(0) = g$ as well as $\tilde{u}'(0) = h$.

5.2. Existence of weak solutions

Let $\{\lambda_j \phi_j\}$ be the eigensystem of $L^2(\Omega)$ described in Theorem 3.6.4, which is also an orthogonal basis of $H_0^1(\Omega)$, see Remark 3.6.7. We want to approximate $\tilde{u}$ by

$$\tilde{u}_m(t) := \sum_{k=1}^m d_m^k(t) \phi_k, \tag{5.2.1}$$

where $\tilde{u}_m$ satisfies

$$\langle \tilde{u}_m''(t), \phi_k \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)} + B[\tilde{u}_m, \phi_k; t] = \langle \tilde{f}(t), \phi_k \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)} \tag{5.2.2}$$

for all $\varphi \in H_0^1(\Omega)$ and a.e. $t \in (0,T)$, and $(\tilde{u}_m(0), \phi_k)_{L^2(\Omega)} = (g, \phi_k)_{L^2(\Omega)}$ as well as $(\tilde{u}_m'(0), \phi_k)_{L^2(\Omega)} = (h, \phi_k)_{L^2(\Omega)}$.

EXERCISE 5.2.1. Show that $d_m^k(t)$ satisfies the linear system of ODE

$$(d_m^k)''(t) + \sum_{\ell=1}^m e^{k\ell}(t) d_m^\ell(t) = f^k(t) \quad \text{for } k, \ell \in \{1, \cdots, m\}.$$

with $e^{k\ell}(t) = B[w_\ell, w_k; t]$ and $f^k(t) := (\tilde{f}(t), \phi_k)_{L^2(\Omega)}$.

By using the standard theory for ODE [**HS99**], there exists a unique C^2 function $\boldsymbol{d}_m(t) = (d_m^1(t), \cdots, d_m^m(t))$ satisfies the ODE described in Exercise 5.2.1, and thus we constructed the function $\tilde{u}_m$ as in (5.2.1), which is called the *Galerkin appriximation* of u. The next step is to obtain an energy estimate for $\tilde{u}_m$.

THEOREM 5.2.2. *If $f \in L^2(0,T;L^2(\Omega))$, then there exists a constant $C = C(\Omega,T)$ such that*

$$\begin{aligned} &\sup_{0 \le t \le T} \left(\|\tilde{u}_m(t)\|_{H_0^1(\Omega)} + \|\tilde{u}_m'(t)\|_{H_0^1(\Omega)} \right) + \|\tilde{u}_m''\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\quad \le C \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{H_0^1(\Omega)} + \|h\|_{L^2(\Omega)} \right) \end{aligned}$$

for all $m \in \mathbb{N}$.

PROOF. The constant $C > 0$ may differ in each line. We will only tracking its dependence. By choosing $\varphi = \tilde{u}_m(t)$ in (5.2.2), we reach

$$\begin{aligned}(\tilde{f}(t), \tilde{u}_m')_{L^2(\Omega)} &= \langle \tilde{u}_m''(t), \tilde{u}_m'(t)\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)} + B[\tilde{u}_m, \tilde{u}_m'; t] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{u}_m'(t)\|_{L^2(\Omega)}^2\right) + B[\tilde{u}_m, \tilde{u}_m'; t].\end{aligned}$$

Furthermore, we can write

$$\begin{aligned}B[\tilde{u}_m, \tilde{u}_m'; t] &= (\nabla\tilde{u}_m, \nabla\tilde{u}_m')_{L^2(\Omega)} + (\boldsymbol{b}\cdot\nabla\tilde{u}_m, \tilde{u}_m')_{L^2(\Omega)} + c(\tilde{u}_m, \tilde{u}_m') \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\tilde{u}_m(t)\|_{L^2(\Omega)}^2\right) + (\boldsymbol{b}\cdot\nabla\tilde{u}_m, \tilde{u}_m')_{L^2(\Omega)} + c(\tilde{u}_m, \tilde{u}_m') \\ &\ge \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\tilde{u}_m(t)\|_{L^2(\Omega)}^2\right) - C\left(\|\tilde{u}_m\|_{H_0^1(\Omega)}^2 + \|\tilde{u}_m'\|_{L^2(\Omega)}^2\right).\end{aligned}$$

Combining the above two estimates, together with the Poincaré inequality (Lemma 3.4.6), we reach

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\tilde{u}_m(t)\|_{L^2(\Omega)}^2 + \|\tilde{u}_m'(t)\|_{L^2(\Omega)}^2\right) \\ &\quad\le C\left(\|\nabla\tilde{u}_m(t)\|_{L^2(\Omega)}^2 + \|\tilde{u}_m'\|_{L^2(\Omega)}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2\right).\end{aligned}$$

For convenience, we write $\eta(t) := \|\nabla\tilde{u}_m(t)\|_{L^2(\Omega)}^2 + \|\tilde{u}_m'(t)\|_{L^2(\Omega)}^2$, and the above inequality reads

$$\eta'(t) \le C(\eta(t) + \|f\|_{L^2(0,T;L^2(\Omega))}^2).$$

By consider the integral factor e^{-Ct}, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(e^{-Ct}\eta(t)) &= -Ce^{-Ct}\eta(t) + e^{-Ct}\eta'(t) \\ &\le Ce^{-Ct}\|f\|_{L^2(0,T;L^2(\Omega))}^2 \le C\|f\|_{L^2(0,T;L^2(\Omega))}^2,\end{aligned}$$

therefore we immediately sees that

$$e^{-C\tau}\eta(\tau) - \eta(0) = \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}(e^{-Ct}\eta(t))\,\mathrm{d}\tau \le C\tau\|f\|_{L^2(0,T;L^2(\Omega))}^2, \quad \text{for a.e. } \tau\in(0,T).$$

However, from Remark 3.6.7 one has

$$\begin{aligned}\eta(0) &= \|\nabla\tilde{u}_m(0)\|_{L^2(\Omega)}^2 + \|\tilde{u}_m'(0)\|_{L^2(\Omega)}^2 \\ &\le \|\tilde{u}_m(0)\|_{H_0^1(\Omega)}^2 + \|\tilde{u}_m'(0)\|_{L^2(\Omega)}^2 \\ &= \sum_{k=1}^m \lambda_j^2|(\tilde{u}_j(0), \phi_j)_{L^2(\Omega)}|^2 + \sum_{k=1}^m |(\tilde{u}_j'(0), \phi_j)_{L^2(\Omega)}|^2 \\ &= \sum_{k=1}^m \lambda_j^2|(g, \phi_j)_{L^2(\Omega)}|^2 + \sum_{k=1}^m |(h, \phi_j)_{L^2(\Omega)}|^2 \\ &\le \sum_{k=1}^\infty \lambda_j^2|(g, \phi_j)_{L^2(\Omega)}|^2 + \sum_{k=1}^\infty |(h, \phi_j)_{L^2(\Omega)}|^2 \\ &= \|g\|_{H_0^1(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2, \qquad (5.2.3)\end{aligned}$$

which proves

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left(\|\tilde{u}_m(t)\|_{H_0^1(\Omega)}+\|\tilde{u}_m'(t)\|_{H_0^1(\Omega)}\right)\\
&\qquad\leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))}+\|g\|_{H_0^1(\Omega)}+\|h\|_{L^2(\Omega)}\right).
\end{aligned}\tag{5.2.4}$$

In order to estimate the term $\tilde{u}_m''(t)$, we fix any $v\in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)}\leq 1$ and it can be uniquely decomposed as $v=v^1+v^2$, where $v^1\in\operatorname{span}\{\phi_k\}_{k=1}^m$ and $(v^2,w_k)_{L^2(\Omega)}=0$ for all $k=1,\cdots,m$. It is easy to verify that

$$\begin{aligned}
&\langle\tilde{u}_m''(t),v\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)}=(\tilde{u}_m''(t),v)_{L^2(\Omega)}\\
&\quad=(\tilde{u}_m''(t),v^1)_{L^2(\Omega)}=(\tilde{f}(t),v^1)-B[\tilde{u}_m(t),v^1;t].
\end{aligned}$$

Since $(v^2,w_k)_{L^2(\Omega)}=0$, then by using Remark 3.6.7 one can easily verify that $(v^2,w_k)_{H_0^1(\Omega)}=0$, and thus $\|v^1\|_{H_0^1(\Omega)}\leq 1$. Hence one sees that

$$|\langle\tilde{u}_m''(t),v\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)}|\leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))}+\|\tilde{u}_m(t)\|_{H_0^1(\Omega)}\right)\quad\text{for a.e. } t\in(0,T),$$

which implies

$$\begin{aligned}
\|\tilde{u}_m''(t)\|_{H^{-1}(\Omega)}&=\sup_{\|v\|_{H_0^1(\Omega)}\leq 1}|\langle\tilde{u}_m''(t),v\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)}|\\
&\leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))}+\|\tilde{u}_m(t)\|_{H_0^1(\Omega)}\right)\quad\text{for a.e. } t\in(0,T).
\end{aligned}$$

Consequently, we reach

$$\begin{aligned}
&\int_0^T\|\tilde{u}_m''(t)\|_{H^{-1}(\Omega)}^2\,\mathrm{d}t\\
&\quad\leq C\int_0^T\left(\|f\|_{L^2(0,T;L^2(\Omega))}^2+\|\tilde{u}_m(t)\|_{H_0^1(\Omega)}^2\right)\mathrm{d}t\\
&\quad\leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))}+\|g\|_{H_0^1(\Omega)}+\|h\|_{L^2(\Omega)}\right),
\end{aligned}$$

where the last inequality follows from (5.2.4). Finally, combining this with (5.2.4), we conclude our lemma. □

EXERCISE 5.2.3 (Gronwall inequality). Let $t\mapsto\eta(t)$ be a non-negative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality

$$\eta'(t)\leq\phi(t)\eta(t)+\psi(t)$$

where $\phi(t)$ and $\psi(t)$ are non-negative integrable functions on $[0,T]$. Show that

$$\eta(t)\leq\exp\left(\int_0^t\phi(s)\,\mathrm{d}s\right)\left[\eta(0)+\int_0^t\psi(s)\,\mathrm{d}s\right]\quad\text{for all } 0\leq t\leq T.$$

In particular, if $\eta'\leq\phi\eta$ on $[0,T]$ and $\eta(0)=0$, show that $\eta\equiv 0$ on $[0,T]$.

EXERCISE 5.2.4 (Gronwall inequality). Let $\xi(t)$ be a nonnegative integrable function on $[0,T]$ which satisfies for a.e. t the integral inequality

$$\xi(t)\leq C_1\int_0^t\xi(s)\,\mathrm{d}s+C_2$$

for some constants $C_1, C_2 \geq 0$. Show that

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t}) \quad \text{for a.e. } 0 \leq t \leq T.$$

In particular, if $\xi(t) \leq C_1 \int_0^t \xi(s)\,\mathrm{d}s$ for a.e. $0 \leq t \leq T$, then $\xi \equiv 0$ on $[0, T]$.

It is important to notice that the upper bound in Theorem 5.2.2 is independent of m. We will recall a standard tool for proving the existence of weak solutions, which is related to Banach-Alaoglu-Bourbaki theorem [**Bre11**, Theorem 3.16] as well as Kakutani theorem [**Bre11**, Theorem 3.17]:

THEOREM 5.2.5 ([**Bre11**, Proposition 3.5 and Theorem 3.18]). *Let X be a reflexive (roughly speaking, $X^{**} \cong X$ with respect to some suitable topology) Banach space and let $\{u_m\}$ be a bounded sequence in X. Then there exists a subsequence $\{u_{m_\ell}\}$ that converges weakly to some $u \in X$ in the sense of*

$$\langle f, u_{m_\ell} \rangle_{X^* \otimes X} \to \langle f, u \rangle_{X^* \otimes X} \quad \text{as } \ell \to \infty.$$

In view of the upper bound in Theorem 5.2.2 which is independent of m (in other words, uniform with respect to m), by choosing the space X with norm (**Note.** here we choose L^2 in the time variable, rather than L^∞: the dual of L^1 is L^∞, but the dual of L^∞ is BMO, but not L^1)

$$\|\tilde{v}\|_X := \|\tilde{v}\|_{L^2(0,T;H_0^1(\Omega))} + \|\tilde{v}'\|_{L^2(0,T;H_0^1(\Omega))} + \|\tilde{v}''\|_{L^2(0,T;H^{-1}(\Omega))},$$

there exists a subsequence $\{u_{m_\ell}\}_{\ell=1}^\infty$ and $u \in L^2(0, T; H_0^1(\Omega))$ with $\partial_t u \in L^2(0, T; L^2(\Omega))$ and $\partial_t'' u \in L^2(0, T; H^{-1}(\Omega))$ such that

$$\begin{cases} u_{m_\ell} \to u & \text{weakly in } L^2(0, T; H_0^1(\Omega)), \\ \partial_t u_{m_\ell} \to \partial_t u & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \partial_t^2 u_{m_\ell} \to \partial_t^2 u & \text{weakly in } L^2(0, T; H^{-1}(\Omega)). \end{cases} \tag{5.2.5}$$

We now fix an integer N and choose a function $v \in C^\infty([0, T]; H_0^1(\Omega))$ of the form

$$\tilde{v}(t) = \sum_{k=1}^N d^k(t)\phi_k \quad \text{where } d^k \in C^\infty([0, T]). \tag{5.2.6}$$

Then from (5.2.2) we have

$$\int_0^T \langle \tilde{u}_m''(t), \tilde{v}(t) \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)}\,\mathrm{d}t + \int_0^T B[\tilde{u}_m, \tilde{v}; t]\,\mathrm{d}t = \int_0^T (\tilde{f}(t), \tilde{v}(t))_{L^2(\Omega)}\,\mathrm{d}t.$$

Choosing $m = m_\ell$ and from (5.2.5), we have

$$\int_0^T \langle \tilde{u}''(t), \tilde{v}(t) \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)}\,\mathrm{d}t + \int_0^T B[\tilde{u}, \tilde{v}; t]\,\mathrm{d}t = \int_0^T (\tilde{f}(t), \tilde{v}(t))_{L^2(\Omega)}\,\mathrm{d}t. \tag{5.2.7}$$

By using the density lemma (Corollary 3.3.15) and the eigendecomposition of Laplacian (Theorem 3.6.4), we know that each function in $L^2(0, T; H_0^1(\Omega))$ can be approximated by the functions in the form of (5.2.6). Therefore (5.2.7) is actually valid for all

$$v \in L^2(0, T; H_0^1(\Omega)).$$

By choosing the test function v which is independent of t in (5.2.7), we immediately see that the limiting function u satisfies (5.1.2). If we choose $v \in C^\infty([0,T]; H_0^1(\Omega))$ with $\tilde{v}(T) = \tilde{v}'(T) = 0$ in (5.2.7), integration by parts twice with respect to t we find

$$\begin{aligned}&\int_0^T (\tilde{v}''(t), \tilde{u}(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T B[\tilde{u}, \tilde{v}; t]\,\mathrm{d}t \\ &\quad= \int_0^T (\tilde{f}(t), \tilde{v}(t))_{L^2(\Omega)}\,\mathrm{d}t - (\tilde{u}(0), \tilde{v}'(0))_{L^2(\Omega)} + \langle \tilde{u}'(0), \tilde{v}(0)\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)}.\end{aligned} \tag{5.2.8}$$

If we choose v be as in the form of (5.2.6), we reach

$$\begin{aligned}&\int_0^T (\tilde{v}''(t), \tilde{u}_m(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T B[\tilde{u}_m, \tilde{v}; t]\,\mathrm{d}t \\ &\quad= \int_0^T (\tilde{f}(t), \tilde{v}(t))_{L^2(\Omega)}\,\mathrm{d}t - (\tilde{u}_m(0), \tilde{v}'(0))_{L^2(\Omega)} + \langle \tilde{u}_m'(0), \tilde{v}(0)\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)} \\ &\quad= \int_0^T (\tilde{f}(t), \tilde{v}(t))_{L^2(\Omega)}\,\mathrm{d}t - (g(0), \tilde{v}'(0))_{L^2(\Omega)} + \langle h(0), \tilde{v}(0)\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)}\end{aligned}$$

because $(\tilde{u}_m(0), \phi_k)_{L^2(\Omega)} = (g, \phi_k)_{L^2(\Omega)}$ as well as $(\tilde{u}_m'(0), \phi_k)_{L^2(\Omega)} = (h, \phi_k)_{L^2(\Omega)}$. Again choosing $m = m_\ell$ and from (5.2.5), from the above equation we see that

$$\begin{aligned}&\int_0^T (\tilde{v}''(t), \tilde{u}(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T B[\tilde{u}, \tilde{v}; t]\,\mathrm{d}t \\ &\quad= \int_0^T (\tilde{f}(t), \tilde{v}(t))_{L^2(\Omega)}\,\mathrm{d}t - (g(0), \tilde{v}'(0))_{L^2(\Omega)} + \langle h(0), \tilde{v}(0)\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)}.\end{aligned} \tag{5.2.9}$$

Combining (5.2.8) and (5.2.9), we see that

$$\begin{aligned}&-(\tilde{u}(0), \tilde{v}'(0))_{L^2(\Omega)} + \langle \tilde{u}'(0), \tilde{v}(0)\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)} \\ &\quad= -(g(0), \tilde{v}'(0))_{L^2(\Omega)} + \langle h(0), \tilde{v}(0)\rangle_{H^{-1}(\Omega)\oplus H_0^1(\Omega)}.\end{aligned}$$

By arbitrariness of $\tilde{v}'(0)$ and $\tilde{v}(0)$, we verify the boundary conditions in Definition 5.1.2, and hence we conclude the following theorem:

THEOREM 5.2.6. *There exists a weak solution (Definition 5.1.2) of the hyperbolic IBVP (5.0.1) satisfies the energy estimate*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|u(t,\cdot)\|_{H_0^1(\Omega)} + \|\partial_t u(t,\cdot)\|_{H_0^1(\Omega)}\right) + \|\partial_t^2 u(t,\cdot)\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\quad\le C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{H_0^1(\Omega)} + \|h\|_{L^2(\Omega)}\right).\end{aligned} \tag{5.2.10}$$

5.3. Uniqueness of weak solutions

We now show that the solution is unique.

THEOREM 5.3.1. *There is at most one weak solution (Definition 5.1.2) of the hyperbolic IBVP (5.0.1).*

PROOF. It is suffices to show that the only weak solution of the hyperbolic IBVP (5.0.1) with $f \equiv g \equiv h \equiv 0$ must be $u \equiv 0$. To verify this, fix $0 \leq s \leq T$ and set

$$\tilde{v}(t) := \begin{cases} \int_t^s \tilde{u}(\tau)\,\mathrm{d}\tau & \text{if } 0 \leq t \leq s, \\ 0 & \text{if } s \leq t \leq T. \end{cases}$$

For each $0 \leq t \leq T$ one sees that $\tilde{v}(t) \in H_0^1(\Omega)$ and so

$$\begin{aligned} 0 &= \int_0^s \left(\langle \tilde{u}'', \tilde{v} \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)} + B[\tilde{u}, \tilde{v}; t] \right) \mathrm{d}t \\ &= \int_0^s \left(-(\tilde{u}', \tilde{v}')_{L^2(\Omega)} + B[\tilde{u}, \tilde{v}; t] \right) \mathrm{d}t \quad (\text{since } \tilde{u}'(0) = h \equiv 0 \text{ and } \tilde{v}(s) = 0) \\ &= \int_0^s \left((\tilde{u}', \tilde{u})_{L^2(\Omega)} - B[\tilde{v}', \tilde{v}; t] \right) \mathrm{d}t \quad (\text{since } \tilde{v}' = -\tilde{u}) \\ &= \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \|\tilde{u}(t)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\nabla \tilde{v}(t)\|_{L^2(\Omega)}^2 - \frac{c}{2} \|\tilde{v}(t)\|_{L^2(\Omega)}^2 \right) \mathrm{d}t \\ &\qquad - \int_0^s (\boldsymbol{b} \cdot \nabla \tilde{v}', \tilde{v})_{L^2(\Omega)} \,\mathrm{d}t \\ &= \frac{1}{2} \|\tilde{u}(s)\|_{L^2(\Omega)}^2 - \frac{1}{2} \overbrace{\|\nabla \tilde{v}(s)\|_{L^2(\Omega)}^2}^{=0} - \frac{c}{2} \overbrace{\|\tilde{v}(s)\|_{L^2(\Omega)}^2}^{=0} \\ &\qquad - \frac{1}{2} \| \overbrace{\tilde{u}(0)}^{=g\equiv 0} \|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \tilde{v}(0)\|_{L^2(\Omega)}^2 + \frac{c}{2} \|\tilde{v}(0)\|_{L^2(\Omega)}^2 \\ &\qquad - \overbrace{(\boldsymbol{b} \cdot \nabla \tilde{v}(s), \tilde{v}(s))_{L^2(\Omega)}}^{=0} + (\boldsymbol{b} \cdot \nabla \tilde{v}(0), \tilde{v}(0))_{L^2(\Omega)} + \int_0^s (\boldsymbol{b} \cdot \nabla \tilde{v}, \tilde{v}')_{L^2(\Omega)} \,\mathrm{d}t, \end{aligned}$$

Then we have

$$\begin{aligned} &\|\tilde{u}(s)\|_{L^2(\Omega)}^2 + \|\nabla \tilde{v}(0)\|_{L^2(\Omega)}^2 \\ &\quad = -c\|\tilde{v}(0)\|_{L^2(\Omega)}^2 - (\boldsymbol{b} \cdot \nabla \tilde{v}(0), \tilde{v}(0))_{L^2(\Omega)} - \int_0^s (\boldsymbol{b} \cdot \nabla \tilde{v}, \tilde{v}')_{L^2(\Omega)} \,\mathrm{d}t \\ &\quad = -c\|\tilde{v}(0)\|_{L^2(\Omega)}^2 - (\boldsymbol{b} \cdot \nabla \tilde{v}(0), \tilde{v}(0))_{L^2(\Omega)} - \int_0^s (\boldsymbol{b} \cdot \nabla \tilde{v}, \tilde{u})_{L^2(\Omega)} \,\mathrm{d}t \quad (\text{since } \tilde{v}' = -\tilde{u}) \end{aligned}$$

It is not difficult to see that

$$|(\boldsymbol{b} \cdot \nabla \tilde{v}(0), \tilde{v}(0))_{L^2(\Omega)}| \leq \frac{1}{2} \|\nabla \tilde{v}(0)\|_{L^2(\Omega)}^2 + C\|\tilde{v}(0)\|_{L^2(\Omega)}^2$$

The constant $C > 0$ may differ in each line. Hence we see that

$$\begin{aligned} &\|\tilde{u}(s)\|_{L^2(\Omega)}^2 + \|\nabla \tilde{v}(0)\|_{L^2(\Omega)}^2 \\ &\quad \leq C \int_0^s \left(\|\nabla \tilde{v}(t)\|_{L^2(\Omega)}^2 + \|\tilde{u}(t)\|_{L^2(\Omega)}^2 \right) \mathrm{d}t + C\|\tilde{v}(0)\|_{L^2(\Omega)}^2. \end{aligned} \tag{5.3.1}$$

Now we write

$$\tilde{w}(t) := \int_0^t \tilde{u}(\tau)\,\mathrm{d}\tau,$$

and now (5.3.1) reads

$$\begin{aligned}
&\|\tilde{u}(s)\|^2_{L^2(\Omega)} + \|\nabla\tilde{w}(s)\|^2_{L^2(\Omega)} \\
&\quad\le C\int_0^s \Big(\|\nabla(\tilde{w}(t)-\tilde{w}(s))\|^2_{L^2(\Omega)} + \|\tilde{u}(t)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t + C\|\tilde{w}(s)\|^2_{L^2(\Omega)} \\
&\quad\le C\int_0^s \Big(\|\nabla\tilde{w}(t)\|^2_{L^2(\Omega)} + \|\nabla\tilde{w}(s)\|^2_{L^2(\Omega)} + \|\tilde{u}(t)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t
\end{aligned}$$

since $\|\tilde{w}(s)\|_{L^2(\Omega)} \le \int_0^s \|\tilde{u}(t)\|_{L^2(\Omega)}\,\mathrm{d}t$. It is important to observe that C is independent of s, hence we obtain

$$\begin{aligned}
&\|\tilde{u}(s)\|^2_{L^2(\Omega)} + (1-sC)\|\nabla\tilde{w}(s)\|^2_{L^2(\Omega)} \\
&\quad\le C\int_0^s \Big(\|\nabla\tilde{w}(t)\|^2_{L^2(\Omega)} + \|\tilde{u}(t)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t.
\end{aligned}$$

Hence we can choose $T_1 := \frac{1}{2C}$, and so

$$\|\tilde{u}(s)\|^2_{L^2(\Omega)} + \|\nabla\tilde{w}(s)\|^2_{L^2(\Omega)} \le C_0\int_0^s \Big(\|\nabla\tilde{w}(t)\|^2_{L^2(\Omega)} + \|\tilde{u}(t)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t.$$

Consequently, the integral form of Gronwall's inequality (Exercise 5.2.4) implies $\tilde{u}(t) \equiv 0$ for all $0 \le t \le T_1$. We apply the same argument on the intervals $[T_1, 2T_1]$, $[2T_1, 3T_1]$ and so so, finally we conclude $u \equiv 0$, which complete the uniqueness proof. $\square$

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