COMPLEX ANALYSIS (701026001, 112-1) - HOMEWORK 10

Return to TA by: December 26, 2023 (Tuesday) 16:00

Total marks: 50 (10 bonus)

Exercise 1 (10 points)**.** Compute $\operatorname{Log}(1+\mathbf{i})$.

Exercise 2 (10 points)**.** Show that $\operatorname{Log}(1+z) = -\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n}$ for all $z \in B_1$.

Exercise 3 (10 points)**.** Prove that $\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)$ converges to a nonzero limit.

Exercise 4 (10 points)**.** Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of **positive** real numbers. Show that

$$a_1 + a_2 + \cdots + a_N \leq \prod_{k=1}^{N}(1+a_k) \leq e^{a_1+a_2+\cdots+a_N} \quad \text{for all } N \in \mathbb{N}.$$

By using this, show that $\prod_{k=1}^{\infty}(1+a_k)$ converges to a nonzero limit if and only if $\sum_{k=1}^{\infty} a_k$ converges.

Exercise 5 (10 points)**.** Let $a_k := \frac{(-1)^k}{\sqrt{k}}$ for all $k = 2, 3, 4, \cdots$. Show that $\sum_{k=1}^{\infty} a_k$ converges but $\prod_{k=1}^{\infty}(1+a_k)$ diverges to zero.

Exercise 6 (10 points)**.** Let K be a compact set in $\mathbb{C}$, and we consider a continuous function $g : K \to \mathbb{C}$. Show that the set $g(K) := \{g(z) : z \in K\}$ is compact in $\mathbb{C}$.