

## COMPLEX ANALYSIS (701026001, 112-1) - HOMEWORK 1

Return to TA by: September 26, 2023 (Tuesday) 16:00

Total marks: 50 (with 6 bonus marks)

\* *Minor update (September 21, 2023): a typo in Exercise 4 was found and fixed*

**Exercise 1.** (10 points) Show that  $|z| \leq |\Re z| + |\Im z|$  for all  $z \in \mathbb{C}$ . When is equality possible?

**Exercise 2.** (10 points) Determine whether the series  $\sum_{k=1}^{\infty} \frac{i}{k^2+i}$  and  $\sum_{k=1}^{\infty} \frac{1}{k+i}$  converges or not.

**Exercise 3.**

- (a) (5 points) Show that the  $n^{\text{th}}$  roots of 1, aside from 1, satisfy the “cyclotomic” equation  $z^{n-1} + z^{n-2} + \dots + z + 1 = 0$ .
- (b) (5 points) Let  $P(z) = 1 + 2z + 3z^2 + \dots + nz^{n-1}$ . Show that all the zeros of  $P(z)$  are inside the closed unit disc.

**Exercise 4.** (10 points) Let  $P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$  with all  $a_i \in \mathbb{R}$  and  $0 \leq a_0 \leq a_1 \leq \dots \leq a_n$ . Show that all the zeros of  $P(z)$  are inside the closed unit disc.

**Exercise 5.** (10 points) Let  $S = \{x + iy : x = 0\} \cup \{x + iy : x > 0, y = \sin \frac{1}{x}\}$ . Show that  $S$  is topologically connected.

**Exercise 6** (Bonus). Use MATLAB (or other software) to plot the region  $\{\varphi(z) : |z| < 1\}$  when

- (a) (2 points)  $\varphi(z) = \frac{1}{m}z^m$  (choose any two integers  $m \geq 2$  to plot)
- (b) (2 points)  $\varphi(z) = z - \frac{2\sqrt{2}}{3}z^2 + \frac{1}{3}z^3$  (zoom in near the point  $0.4 + i0$ )
- (c) (2 points)  $\varphi(z) = (z - 1)^2 - (1 - \frac{i}{2})(z - 1)^3$  (zoom in near the origin 0)