COMPLEX ANALYSIS (701026001, 112-1) - HOMEWORK 4

Return to TA by: October 17, 2023 (Tuesday) 16:00

Total marks: 50

Exercise 1. (5+5+5+5+5+5 points)

(a) Prove that $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for all $z_1, z_2 \in \mathbb{C}$.

(b) For each $n \in \mathbb{N}$, show that $(\cos\theta + \mathbf{i}\sin\theta)^n = \cos(n\theta) + \mathbf{i}\sin(n\theta)$ for all $\theta \in \mathbb{R}$.

(c) Show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$ for all $\theta \in \mathbb{R}$.

(d) Show the following identities:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2$$

for all $\theta_1, \theta_2 \in \mathbb{R}$.

(e) Show the following product-to-sum identities:

$$\cos\theta_1\cos\theta_2 = \frac{\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)}{2}$$
$$\sin\theta_1\sin\theta_2 = \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2}$$
$$\sin\theta_1\cos\theta_2 = \frac{\sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)}{2}$$

for all $\theta_1, \theta_2 \in \mathbb{R}$. (Note: The sum-to-product identities are trivial consequences of product-to-sum identities by letting $\theta_1 = \frac{\alpha+\beta}{2}$ and $\theta_2 = \frac{\alpha-\beta}{2}$.)

(f) Compute $\cos(\theta_1 + \theta_2 + \theta_3)$ and $\sin(\theta_1 + \theta_2 + \theta_3)$ for all $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$.

Exercise 2. (5+5 points) Show that e^z is entire by verifying the Cauchy Riemann equation. Show that e^z is entire by proving

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

with radius of convergence $R = +\infty$.

Exercise 3. (5+5 points) Show that $\sin z$ is entire by proving

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

with radius of convergence $R = +\infty$. Show that $\cos z$ is entire by finding its power series representation and compute its radius of convergence.