COMPLEX ANALYSIS (701026001, 112-1) - HOMEWORK 5

Return to TA by: October 24, 2023 (Tuesday) 16:00

Total marks: 50 + 10 bonus

Exercise 1 (10 points)**.** Evaluate $\int_{\mathcal{C}} f$ where $f(z) = x^2 + \mathbf{i}y^2$ and $\mathcal{C} = \left[\ z(t) = t^2 + \mathbf{i}t^2 \ \middle|\ 0 \leq t \leq 1\ \right]$. Here we denote $z = x + \mathbf{i}y$.

Exercise 2 (10 points)**.** Let $\{\mathcal{K}^{(k)}\}$ be a sequence of compact sets in $\mathbb{C} \cong \mathbb{R}^2$ such that $\mathcal{K}^{(1)} \supset \mathcal{K}^{(2)} \supset \mathcal{K}^{(3)} \supset \cdots$. Show that $\bigcap_{k\in\mathbb{N}} \mathcal{K}^{(k)} \neq \emptyset$. [Hint: consider the complement of $\mathcal{K}^{(k)}$]

Exercise 3 (10 points)**.** A set S is called *star-like* if there exists a point $\alpha \in S$ such that the line segment connecting α and z is contained in S for all $z \in S$. Show that a star-like region is simply connected.

Exercise 4 (10 points)**.** Let K be a compact set in $\mathbb{C}$ and let F be a topological closed set in $\mathbb{C}$. If $K \cap F = \emptyset$, show that $\operatorname{dist}(K, F) > 0$. On the other hand, construct topological closed sets F_1, F_2 in $\mathbb{C}$ such that $F_1 \cap F_2 = \emptyset$ but $\operatorname{dist}(F_1, F_2) = 0$.

Exercise 5 (10 points)**.** Evaluate $\int_{\mathcal{C}} f$ where $f(z) = 1/z$ and $\mathcal{C} = \left[\ z(t) = \sin t + \mathbf{i}\cos t \ \middle|\ 0 \leq t \leq 2\pi\ \right]$. Which assumption in Cauchy closed curve theorem does not satisfied by this example?

Exercise 6 (5+5 points)**.** Evaluate $\int_0^{\mathbf{i}} e^z \,\mathrm{d}z$ and $\int_{\pi/2}^{\pi/2+\mathbf{i}} \cos 2z \,\mathrm{d}z$. Here the integrals are defined in the proof of Theorem 3.2.9 in the lecture note.