COMPLEX ANALYSIS (701026001, 112-1) - HOMEWORK 6

Return to TA by: October 31, 2023 (Tuesday) 16:00

Total marks: 50

Exercise 1 (10 points, higher order Cauchy integral formula)**.** Let f be an entire function, let $a \in \mathbb{C}$ and let $\mathcal{C} = \left[\ Re^{\mathbf{i}\theta} \,\middle|\, 0 \leq \theta \leq 2\pi\ \right]$ with $R > |a|$. Show that

$$f^{(k)}(a) = \frac{k!}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)}{(z-a)^{k+1}} \, \mathrm{d}z \quad \text{for all } k = 0, 1, 2, \cdots .$$

Exercise 2 (10 points)**.** Suppose f is entire with zeros $a_1, a_2, \cdots, a_N$, that is, $f(a_k) = 0$ for $k = 1, 2, \cdots, N$, and we define

$$g(z) := \frac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_N)} \quad \text{for all } z \in \mathbb{C} \setminus \{a_1, a_2, \cdots, a_N\}.$$

Show that if $\lim_{z \to a_k} g(z)$ exists for all $k = 1, 2, \cdots, N$, then the extension $\tilde{g}$of g defined by

$$\tilde{g}(z) := \begin{cases} g(z) & , z \in \mathbb{C} \setminus \{a_1, a_2, \cdots, a_N\}, \\ \lim\limits_{z \to a_k} g(z) & , z = a_k \text{ for } k = 1, 2, \cdots, N, \end{cases}$$

is also entire.

Exercise 3 (5+5 points)**.** f is called an *odd* function if $f(z) = -f(-z)$ for all $z \in \mathbb{C}$; f is called *even* if $f(z) = f(-z)$ for all $z \in \mathbb{C}$.

(a) Show that an odd entire function has only odd terms in its power series expansion about $z = 0$.

(b) Prove an analogous result for even functions

Exercise 4 (5+5 points)**.** (a) Suppose f is analytic in a connected region and $f' \equiv 0$ there. Show that f is constant.

(b) Assume that f is analytic in a connected region and that at every point, either $f = 0$ or $f' = 0$. Show that f is constant.

Exercise 5 (5+5 points)**.** (a) Find all entire functions $f = u + iv$ with $u(x, y) = x^2 - y^2$.

(b) Show that there are no entire functions $f = u + iv$ with $u(x, y) = x^2 + y^2$.