COMPLEX ANALYSIS (701026001, 112-1) - HOMEWORK 9

Return to TA by: December 19, 2023 (Tuesday) 16:00

Announcement. I added a remark after Theorem 5.2.6, which is helpful in this homework. Total marks: 50 (10 bonus)

Exercise 1 (10 points, higher order Cauchy integral formula)**.** Let f be an entire function, let $a \in \mathbb{C}$ and let $\mathcal{C} = \left[\, Re^{\mathbf{i}\theta} \, \middle| \, 0 \leq \theta \leq 2\pi \, \right]$ with $R > |a|$. Show that

$$f^{(k)}(a) = \frac{k!}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)}{(z-a)^{k+1}} \, \mathrm{d}z \quad \text{for all } k = 0, 1, 2, \cdots$$

is a special case of Cauchy residue theorem (Theorem 5.3.6).

Exercise 2 (10 points, exponential integral)**.** For each $n \in \mathbb{N}$ and $k = 0, 1, 2, \cdots, n$, prove the following exponential integral

$$n^k = \frac{k!}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{e^{nz}}{z^{k+1}} \, \mathrm{d}z$$

for all simple closed (parametrizable continuous piecewise-C^1) curve $\mathcal{C}$ surrounding the origin.

Exercise 3 (10 points)**.** Show that the image of the unit disc minus the origin under $f(z) = \csc(1/z) \equiv 1/(\sin(1/z))$ is dense in the complex plane. [Hint: Show that $\sin(1/z)$ has a essential singularity at $z = 0$]

Exercise 4 (10 points)**.** Show that if f is analytic in $z \neq 0$ and $f(-z) = -f(z)$ for all $z \neq 0$, then all the even terms in its Laurent expansion about 0 are 0.

Exercise 5 (10+10 points)**.** Classify the singularities of $\dfrac{1}{z^4 + z^2}$ and $\dfrac{\exp(1/z^2)}{z-1}$.