COMPLEX ANALYSIS (701026001, 113-1) - HOMEWORK 1

Return to TA by: September 27, 2024 (Friday) 12:00

Total marks: 50 (with 9 bonus marks)

Exercise 1 (10 points)**.** Show that for any two pairs of integers $\{a, b\}$ and $\{c, d\}$, we can find integers u, v with

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2.$$

Exercise 2 (10 points)**.** Show that $|z| \leq |\mathfrak{Re} z| + |\mathfrak{Im} z|$ for all $z \in \mathbb{C}$. When is equality possible?

Exercise 3 (5+5 points)**.** Determine whether the series $\sum_{k=1}^{\infty} \frac{\mathbf{i}^k}{k^2+\mathbf{i}}$ and $\sum_{k=1}^{\infty} \frac{1}{k+\mathbf{i}}$ converges or not.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a function. If $f(z_0) = 0$, then we called such $z_0 \in \mathbb{C}$ a *zero* of f.

Exercise 4 (10 points)**.** Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ with all $a_i \in \mathbb{R}$. Show that $P(z) = 0$ if and only if $P(\overline{z}) = 0$.

Exercise 5 (10 points)**.** Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ with all $a_i \in \mathbb{R}$ and $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$. Show that all the zeros of $P(z)$ are inside the closed unit disc $\overline{B_1} = \{z \in \mathbb{C} : |z| \leq 1\}$. [Hint: Consider the factor $(1 - z)$ and using a contradiction argument]

Exercise 6 (Bonus, 3+3+3 points)**.** Use MATLAB (or other software) to plot the region $\{\varphi(z) : |z| < 1\}$ when

(a) $\varphi(z) = \frac{1}{m} z^m$ (choose any two integers $m \geq 2$ to plot)
(b) $\varphi(z) = z - \frac{2\sqrt{2}}{3} z^2 + \frac{1}{3} z^3$ (zoom in near the point $0.4 + \mathbf{i}0$)
(c) $\varphi(z) = (z - 1)^2 - (1 - \frac{\mathbf{i}}{2})(z - 1)^3$ (zoom in near the origin 0)