COMPLEX ANALYSIS (701026001, 113-1) - HOMEWORK 2

Return to TA by: October 11, 2024 (Friday) 12:00

Total marks: 50

Exercise 1 (10 points)**.** Let $S = S_1 \cup S_2$ where

$$S_1 = \{x + \mathbf{i}y : x = 0\}, \quad S_2 = \left\{x + \mathbf{i}y : x > 0, y = \sin\frac{1}{x}\right\}.$$

Show that S is topologically connected (despite $S_1 \cap S_2 = \emptyset$). [Hint: Note that both S_1 and S_2 are topologically connected. In order to show that S is topologicall connected, we need to show that S_1 (and S_2) cannot be both relative open and relative closed in S. Note that S_1 is closed in $\mathbb{R}^2$, and thus it is relative closed in S. Therefore, one only need to show that S_1 is not relative open in S.]

Exercise 2 (5 points)**.** Let P be a nonconstant polynomial in z. Show that $|P(z)| \to \infty$ as $|z| \to \infty$.

Exercise 3 (10 points)**.** Let $\{\mathcal{K}^{(k)}\}$ be a sequence of compact sets in $\mathbb{C} \cong \mathbb{R}^2$ such that $\mathcal{K}^{(1)} \supset \mathcal{K}^{(2)} \supset \mathcal{K}^{(3)} \supset \cdots$. Show that $\bigcap_{k\in\mathbb{N}} \mathcal{K}^{(k)} \neq \emptyset$. [Hint: consider the complement of $\mathcal{K}^{(k)}$. Here we again remind that the complex plane $\mathbb{C}$ is not compact.]

Exercise 4. We consider a function

$$f : (0,1) \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{q} & ,\text{if } x = \frac{p}{q} \in (0,1) \cap \mathbb{Q}, q > 0, \gcd(p,q) = 1, \\ 0 & \text{if } x \in (0,1) \setminus \mathbb{Q}. \end{cases}$$

(a) **(5 points)** Show that f is not continuous at all $x_1 \in (0,1) \cap \mathbb{Q}$; and
(b) **(10 points)** show that f is continuous at all $x_0 \in (0,1) \setminus \mathbb{Q}$.

[Hint: consider the set of rational number with denominator at most q, that is, $\mathbb{Q}_q := \mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z} \cup \cdots \cup \frac{1}{q}\mathbb{Z}$. The arguments can be simplify by using the notion of limsup/liminf.]

Exercise 5 (10 points)**.** We now consider the function $f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ by

$$f(x_1, x_2) = \frac{x_1 x_2^2}{x_1^2 + x_2^4} \quad \text{for all } \boldsymbol{x} = (x_1, x_2) \neq (0,0).$$

Show that for each straight line $\mathfrak{L}$ in $\mathbb{R}^2$ passing through the origin one has

$$\lim_{\boldsymbol{x}\to\mathbf{0}, \boldsymbol{x}\in\mathfrak{L}} f(\boldsymbol{x}) = 0,$$

but $\lim_{\boldsymbol{x}\to\mathbf{0}} f(\boldsymbol{x})$ does not exist.