COMPLEX ANALYSIS (701026001, 113-1) - HOMEWORK 4

Return to TA by: October 25, 2024 (Friday) 12:00

Total marks: 50 (with 10 bonus points)

Exercise 1 (10 points)**.** Show that the radius of convergence of $\sum_{n=1}^{\infty} nz^n$ is $R = 1$, and the series also diverges for each $z \in \mathbb{C}$ with $|z| = 1$.

Exercise 2 (10 points)**.** Show that the radius of convergence of $\sum_{n=1}^{\infty} n^{-2} z^n$ is $R = 1$, and the series also converges for each $z \in \mathbb{C}$ with $|z| = 1$.

Exercise 3 (10 points)**.** Show that the radius of convergence of $\sum_{n=1}^{\infty} n^{-1} z^n$ is $R = 1$. In addition, show that the series converges for all $z \in \partial B_1 \setminus \{1\}$ but diverges at $z = 1$.

Exercise 4 (10 points)**.** Show that if $f(z) = \sum_{n=0}^{\infty} C_n z^n$ has a nonzero radius of convergence, then

$$C_n = \frac{f^{(n)}(0)}{n!} \quad \text{for all } n = 0, 1, 2, \cdots ,$$

where $f^{(n)}$ is the n^{th} (complex) derivative of f. Show that for each $n = 0, 1, 2, \cdots$ that

$$f^{(n)}(z) = n! C_n + (n+1)! C_{n+1} z + \frac{(n+2)!}{2!} C_{n+2} z^2 + \cdots$$

for all z in the domain of convergence.

Exercise 5 (10 points)**.** Show that if $f(z)$ and $\overline{f(z)}$ are analytic in a connected open set $\Omega \subset \mathbb{C}$, then there exists $c \in \mathbb{C}$ such that $f(z) = c$ for all $z \in \mathbb{C}$.

Exercise 6 (10 points)**.** If $f(z) = u + \mathbf{i}v$ is a complex function such that u and v are both harmonic, is $f(z)$ necessarily analytic? (**Note.** It is interesting to compare this exercise with Remark 2.1.11)