COMPLEX ANALYSIS (701026001, 113-1) - HOMEWORK 6

Return to TA by: November 8, 2024 (Friday) 12:00

Total marks: 50 (with 10 bonus points)

Exercise 1 (10 points). Let $\{\mathcal{K}^{(k)}\}$ be a sequence of compact sets in $\mathbb{C} \cong \mathbb{R}^2$ such that $\mathcal{K}^{(1)} \supset \mathcal{K}^{(2)} \supset \mathcal{K}^{(3)} \supset \cdots$. Show that $\bigcap_{k\in\mathbb{N}} \mathcal{K}^{(k)} \neq \emptyset$. [Hint: consider the complement of $\mathcal{K}^{(k)}$.]

Exercise 2 (10 points). Let K be a compact set in $\mathbb{C}$ and let F be a topological closed set in $\mathbb{C}$. If $K \cap F = \emptyset$, show that $\operatorname{dist}(K, F) > 0$. On the other hand, construct topological closed sets F_1, F_2 in $\mathbb{C}$ such that $F_1 \cap F_2 = \emptyset$ but $\operatorname{dist}(F_1, F_2) = 0$.

Exercise 3 (10 points). Let Ω be a simply connected open set in $\mathbb{C} \cong \mathbb{R}^2$, and let $u \in C^2(\Omega)$ be a given *real-valued* harmonic function (i.e. $\partial_x^2 u + \partial_y^2 u = 0$ in Ω). Show that there exists a real-valued function $v \in C^2(\Omega)$ such that $F = u + \mathbf{i}v$ is analytic in Ω. [Hint: First define $f := \partial_x u - \mathbf{i}\partial_y u$ and the consider its antiderivative F as in the fundamental theorem of antiderivative. Show that $\Re F = u$.]

Exercise 4 (Higher order Cauchy integral formula for entire functions, 10 points). Let f be an entire function, let $a \in \mathbb{C}$ and let $\mathcal{C} = \left[\ Re^{\mathbf{i}\theta} \;\middle|\; 0 \leq \theta \leq 2\pi\ \right]$ with $R > |a|$. Show that

$$f^{(k)}(a) = \frac{k!}{2\pi\mathbf{i}} \int_{\mathcal{C}} \frac{f(z)}{(z-a)^{k+1}}\,\mathrm{d}z \quad \text{for all } k = 0, 1, 2, \cdots$$

Exercise 5 (10 points). Suppose f is entire and $|f(z)| \leq A + B|z|^{\frac{3}{2}}$ for all $z \in \mathbb{C}$. Show that f is linear polynomial.

Exercise 6 (10 points). Suppose f is entire and $|f'(z)| \leq |z|$ for all $z \in \mathbb{C}$. Show that $f(z) = a + bz^2$ with $|b| \leq \frac{1}{2}$.