FOURIER ANALYSIS (MATS315) - EXERCISE 1

Return by: September 16, 2022 (Friday)

Exercise 1 (10 pts)**.** Determine all the values of $a \in \mathbb{R}$ for which the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & \text{for all } x \geq a, \\ 0 & \text{for all } x < a, \end{cases}$$

has weak derivative.

Exercise 2 (10 pts)**.** Compute the weak derivative of order one of $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) := \operatorname{sgn}(x)\sqrt{|x|}$ for every $x \in (-1, 1)$, where $\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$

Exercise 3 (10 pts)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := x|y|$ for each $(x, y) \in \mathbb{R}^2$. Prove that the weak derivative $\partial_1^2 \partial_2 f$ exists, while the weak derivative $\partial_1 \partial_2^2 f$ does not.

Exercise 4 (10 pts)**.** Let $\epsilon \in (0, 1)$ and consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} |x|^{-\epsilon} & \text{if } x \in \mathbb{R}^n \setminus \{0\}, \\ 1 & \text{if } x = 0. \end{cases}$$

Prove that $\partial_j f$ exists in the weak sense for each $j \in \{1, \cdots, n\}$ if and only if $n \geq 2$. Also compute the weak derivatives $\partial_j f$ for all $j = 1, \cdots, n$ in the case when $n \geq 2$.

Exercise 5 (10 pts)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := H(x) + H(y)$ for each $(x, y) \in \mathbb{R}^2$. Prove that for multi-indices $\alpha = (1, 1)$ and $\beta = (1, 0)$ the weak derivatives $\partial^\alpha f$ and $\partial^{\alpha+\beta} f$ exists, while the weak derivative $\partial^\beta f$ does not exist.

Exercise 6 (10 pts)**.** Let $f(x) = x^2$ for $x \in (-\pi, \pi)$. Compute its Fourier series and find the sum of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$.

Exercise 7 (10 pts)**.** Put $x = \frac{\pi}{4}$ in the Fourier sine series of the constant function $f(x) = 1$ for $x \in (0, \pi)$ to compute the sum

$$\left(1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots\right) + \left(\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \cdots\right) = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots .$$

(Note: The left-hand-side cannot be arbitrarily rearranged because they are only conditionally, not absolutely, convergent.)

Exercise 8 (10 pts)**.** Compute the Fourier series of $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2 - 1}.$$

Exercise 9 (10 pts)**.** Compute the Fourier series of e^x on $(-\pi, \pi)$.

Exercise 10 (10 pts)**.** Show how the Fourier series on $(-\ell, \ell)$ can be derived from the series on $(-\pi, \pi)$ by changing variables $y = \frac{\pi}{\ell}x$.