## FOURIER ANALYSIS (MATS315) - EXERCISE 2

Return by: September 23, 2022 (Friday)

**Exercise 1** (Exercise 1.5.2, 10 pts)**.** Prove Theorem 1.5.1.

**Exercise 2** (Exercise 1.7.3, 10 pts)**.** Verify that the Fejér kernel is an approximate identity as in Definition 1.3.6.

**Exercise 3** (Exercise 2.1.4, Riemann-Lebesgue, 10 pts)**.** Prove that if $f \in L^1(\mathbb{R}^n)$, then $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$. [Hint: $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, and consider the Laplacian.]

**Exercise 4** (Exercise 2.2.3, 10 pts)**.** Prove that for each fixed number $a \in (0, \infty)$ the function $f(x) = e^{-a|x|^2}$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{S}(\mathbb{R}^n)$. Therefore $C_c^\infty(\mathbb{R}^n) \subsetneq \mathscr{S}(\mathbb{R}^n) \subsetneq C^\infty(\mathbb{R}^n)$. (Note: $e^{-|x|}$ is not in Schwartz space since it is not $C^\infty$ near the origin.)

**Exercise 5** (Exercise 2.2.4, 10 pts)**.** Verify that the function $\mathsf{d}_{\mathscr{S}(\mathbb{R}^n)}$ given in (2.2.5) is a metric. [Hint: If $\|\cdot\|$ is a norm on a vector space, show hat $\frac{\|u+v\|}{1+\|u+v\|} \leq \frac{\|u\|}{1+\|u\|} + \frac{\|v\|}{1+\|v\|}$.]

**Exercise 6** (Exercise 2.2.9, 10 pts)**.** Prove that for each $s \in \mathbb{R}$ the function $f(x) := \langle x \rangle^s$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n)$.

**Exercise 7** (Exercise 2.2.10, 10 pts)**.** Prove that the function $f(x) := e^{i|x|^2}$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n)$.