FOURIER ANALYSIS (MATS315) - EXERCISE 4

Return by: October 07, 2022 (Friday)

Exercise 1 (Exercise 2.7.10, 10 pts)**.** Prove Lemma 2.7.9.

Exercise 2 (Exercise 2.8.17, 10 pts)**.** Prove that for every $c \in \mathbb{R}$ one has

$$(e^{-c|x|})' = -ce^{-cx}H(x) + ce^{cx}H(-x) \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

Exercise 3 (Exercise 2.8.18, 10 pts)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \ln |x| - x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that f is a continuous function and compute its distributional derivative (of order 1) f'.

Exercise 4 (Exercise 2.8.19, 10 pts)**.** Let $n = 1$ an $T = \sum_{j=1}^{\infty} \partial^j \delta_j \in \mathscr{D}'(\mathbb{R})$, that is,

$$T(\varphi) = \sum_{j=1}^{\infty} (-1)^j \varphi^{(j)}\Big|_{x=j} \quad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}).$$

Show that T does not have finite order.

Exercise 5 (Exercise 2.9.5, 10 pts)**.** Show that if $A \subset \mathbb{R}^n$ is compact and $B \subset \mathbb{R}^n$ is closed, then $A + B$ is a closed set in $\mathbb{R}^n$.

Exercise 6 (Exercise 2.11.4, 10 pts)**.** Let $T_1 = 1$, $T_2 = \delta_0'$ and $T_3 = H$ (the Heaviside unit step function) and show that

$$(T_1 * T_2) * T_3 \text{ and } (T_2 * T_3) * T_1 \text{ both exist but they are not identical.}$$

This exercise emphasizes that in general the associativity property in Proposition 2.11.3(1) only valid with the condition on supports.