GEOMETRY (701939001, 751764001, 113-2) - HOMEWORK 5

Return to TA by: May 13, 2025 (Tuesday) 16:00

Total marks: 50

Exercise 1 (10 points)**.** Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular C^3-curve which is parameterized using arc length $s \in I$ such that $\kappa(s) > 0$ for all $s \in I$. Show that the torsion τ is given by

$$\tau(s) = -\frac{(\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)) \cdot \boldsymbol{\alpha}'''(s)}{|\kappa(s)|^2} \quad \text{for all } s \in I.$$

Exercise 2. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular C^3-curve (not necessarily parameterized by arc length) such that $\kappa(s) > 0$ for all $t \in I$. Let $s(t)$ be its arc length from some point $t_0 \in I$, and let $t = t(s)$ be its inverse function and set $\boldsymbol{\alpha}'(t) := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha}(t)$, $\boldsymbol{\alpha}''(t) := (\frac{\mathrm{d}}{\mathrm{d}t})^2\boldsymbol{\alpha}(t)$ and $\boldsymbol{\alpha}'''(t) := (\frac{\mathrm{d}}{\mathrm{d}t})^3\boldsymbol{\alpha}(t)$.

(a) (10+10 points)**.** Show that $\frac{\mathrm{d}}{\mathrm{d}s}t(s) = |\boldsymbol{\alpha}'(t(s))|^{-1}$ and $(\frac{\mathrm{d}}{\mathrm{d}s})^2 t(s) = -|\boldsymbol{\alpha}'(t(s))|^{-4}\boldsymbol{\alpha}'(t(s)) \cdot \boldsymbol{\alpha}''(t(s))$.

(b) (10 points)**.** Show that the curvature κ is given by

$$\kappa(t) = \frac{|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)|}{|\boldsymbol{\alpha}'(t)|^3} \quad \text{for all } t \in I.$$

(c) (10 points)**.** Show that the torsion τ is given by

$$\tau(t) = -\frac{(\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)) \cdot \boldsymbol{\alpha}'''(t)}{|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)|^2} \quad \text{for all } t \in I.$$