GEOMETRY (701939001, 751764001, 113-2) - HOMEWORK 7

Return to TA by: May 27, 2025 (Tuesday) 16:00

Total marks: 50

Exercise 1 (10 points)**.** Let S be a regular surface and let $\boldsymbol{x} : U \to V \cap S$ be a local coordinates near $\boldsymbol{p} = \boldsymbol{x}(\boldsymbol{q})$. Show that $\mathrm{d}\boldsymbol{x}_{\boldsymbol{q}} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective if and only if $\partial_u \boldsymbol{x}(u,v) \times \partial_v \boldsymbol{x}(u,v) \neq 0$.

Exercise 2 (10 points)**.** Show that the unit sphere $\mathbb{S}^2$ is a regular surface which can be covered by the local charts $\{\boldsymbol{x}_i, U_i\}_{i=1}^4$ given by the *local spherical coordinate*:

$$\begin{cases} \boldsymbol{x}_1(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) & ,(\theta,\varphi) \in U_1 = (0,\pi) \times (0,2\pi), \\ \boldsymbol{x}_2(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) & ,(\theta,\varphi) \in U_2 = (0,\pi) \times (-\pi,\pi), \\ \boldsymbol{x}_3(\theta,\varphi) = (\sin\theta\cos\varphi, \cos\theta, \sin\theta\sin\varphi) & ,(\theta,\varphi) \in U_3 = (0,\pi) \times (0,2\pi), \\ \boldsymbol{x}_4(\theta,\varphi) = (\sin\theta\cos\varphi, \cos\theta, \sin\theta\sin\varphi) & ,(\theta,\varphi) \in U_4 = (0,\pi) \times (-\pi,\pi). \end{cases}$$

Exercise 3 (10 points)**.** The stereographic projection $\pi : \mathbb{S}^2 \setminus \{\boldsymbol{e}_3\} \to \mathbb{R}^2$, where $\boldsymbol{e}_3 = (0,0,1)$ is the north pole, carries a point $\boldsymbol{p} \in \mathbb{S}^2 \setminus \{\boldsymbol{e}_3\}$ onto the intersection of the xy plane with the straight line which connects $\boldsymbol{e}_3$ to $\boldsymbol{p}$, that is,

$$\pi(\boldsymbol{p}) := (\boldsymbol{e}_3 + [\boldsymbol{p} - \boldsymbol{e}_3])|_{z=0}.$$

Compute the formula of $\pi^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{\boldsymbol{e}_3\}$ and use this to show that $\pi^{-1}(\mathbb{R}^2) = \mathbb{S}^2 \setminus \{\boldsymbol{e}_3\}$ is a regular surface. From this, one immediately sees that the unit sphere $\mathbb{S}^2$ is a regular surface which can be covered by the local charts $\{\boldsymbol{x}_i, U_i\}_{i=1}^4$ given by

$$\begin{cases} \boldsymbol{x}_1 = \pi^{-1} & ,U_1 = \mathbb{R}^2, \\ \boldsymbol{x}_1 = -\pi^{-1} & ,U_2 = \mathbb{R}^2. \end{cases}$$

Exercise 4 (10 points)**.** Show that it is not possible to cover $\mathbb{S}^2$ by just a single chart. (**Hint.** Using a contradiction argument)

Exercise 5 (10 points)**.** Show that the torus $\left\{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - 1)^2 + z^2 = \frac{1}{4}\right\}$ is a regular surface