DIFFERENTIAL EQUATIONS (751873001, 113-1) - HOMEWORK 4

Return by November 7, 2024 (Thursday) 23:59

Total marks: 50 (with 10 bonus points)

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10 points)**.** For each $A \in \mathbb{C}^{n\times n}$, show that $\det(\exp(A)) = e^{\mathrm{tr}\,(A)}$. In addition, show that $\mathrm{tr}\,(A) = \lambda_1 + \cdots + \lambda_n$, where $\lambda_j \in \mathbb{C}$ are eigenvalues (may identical) of A.

Exercise 2 (5+5+5 points)**.** Let

$$A_1 = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Compute $\exp(A_1)$, $\exp(A_2)$ and $\exp(A_3)$.

Exercise 3 (5 points)**.** Show that for any $a,b,d \in \mathbb{C}$ that

$$\exp\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} e^a & b\frac{e^a - e^d}{a-d} \\ 0 & e^d \end{pmatrix}.$$

Since

$$\lim_{a\to d} \frac{e^a - e^d}{a-d} = e^a,$$

we simply interpret $\frac{e^a-e^d}{a-d}$ as e^a when $d = a$. (**Hint.** Show that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^m = \begin{pmatrix} a^m & b\frac{a^m - d^m}{a-d} \\ 0 & b^m \end{pmatrix}$$

for all $m \in \mathbb{N}$ and $a \neq d$.)

Exercise 4 (10 points)**.** Let $0 < s < 1$, by using the integration by parts on $\Gamma(1-s)$, where Γ is the gamma function, show that

$$\lambda^s = \frac{1}{|\Gamma(-s)|}\int_0^\infty (1 - e^{t\lambda})t^{-1-s}\,\mathrm{d}t \quad \text{for all } \lambda > 0.$$

Exercise 5 (5+5 points)**.** Show that:

(a) If A is unipotent, then $\exp(\log(A)) = A$.
(b) If B is nilpotent, then $\log(\exp(B)) = B$.

(**Hint.** Let $A(t) := I + t(A - I)$ and show that $\exp(\log(A(t)))$ depends polynomially on t and that $\exp(\log(A(t))) = A(t)$ for all sufficiently small t)

Exercise 6 (10 points)**.** Show that there exists a constant $c > 0$ such that

$$\|\log(I + A) - A\| \leq c\|A\|^2$$

holds true for all $A \in \mathbb{C}^{n\times n}$ with $\|A\| \leq 1/2$.