DIFFERENTIAL EQUATIONS (751873001, 114-1) - HOMEWORK 4

Return by October 17, 2025 (Friday) 23:59

Total marks: 50

Special requirement. All homeworks must be prepared by using LaTeX.

Exercise 1 (10+10 points)**.** Show that:

(a) If A is unipotent, then $\exp(\log(A)) = A$.
(b) If B is nilpotent, then $\log(\exp(B)) = B$.

(**Hint.** Let $A(t) := I + t(A - I)$ and show that $\exp(\log(A(t)))$ depends polynomially on t and that $\exp(\log(A(t))) = A(t)$ for all sufficiently small t)

Exercise 2 (10 points)**.** Show that there exists a constant $c > 0$ such that

$$\|\log(I + A) - A\| \leq c\|A\|^2$$

holds true for all $A \in \mathbb{C}^{n\times n}$ with $\|A\| \leq 1/2$.

Exercise 3 (10 points)**.** Let $n \in \mathbb{N}$ and we define $\mathrm{Sp}(2n,\mathbb{C}) := \{M \in \mathbb{C}^{2n\times 2n} : M^\intercal \Omega M = \Omega\}$, where $\Omega \in \mathbb{C}^{2n\times 2n}$ is given by

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (I \text{ is the } n\times n \text{ identity matrix}).$$

Show that $\mathrm{Sp}(2n,\mathbb{C})$ is a subgroup of $\mathrm{SL}(2n,\mathbb{C})$, i.e. $\mathrm{Sp}(2n,\mathbb{C}) \subset \mathrm{SL}(2n,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ itself forms a group with respect to matrix multiplication.

Remark. $\mathrm{Sp}(2n,\mathbb{C})$ is called the symplectic Lie group.

Exercise 4 (10 points)**.** Characterize the corresponding lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ of $\mathrm{Sp}(2n,\mathbb{C})$, called the symplectic Lie algebra, in the sense of Definition 3.1.57.