

DIFFERENTIAL EQUATIONS (751873002, 114-2) - HOMEWORK 1

Return by March 20, 2026 (Friday) 23:59

Total marks: 50

Special requirement. All homeworks must be prepared by using L^AT_EX.

Exercise 1 (10 points). For each $1 < p < \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1$, show that the following inequality:

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \quad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

Exercise 2 (10 points). Let Ω be an open set in \mathbb{R}^n . Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1$. Show that $fg \in L^1(\Omega)$ and the following Hölder's inequality holds:

$$\int_{\Omega} |f(\mathbf{x})g(\mathbf{x})| \, d\mathbf{x} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}$$

and the equality holds if and only if there exists $c \in \mathbb{R}$ such that $|g(\mathbf{x})| = c|f(\mathbf{x})|^{p-1}$ for a.e. $\mathbf{x} \in \Omega$.

Exercise 3 (10 points). Let Ω be an open set in \mathbb{R}^n , show that $\|\cdot\|_{L^p(\Omega)}$ defines a norm for each $1 \leq p \leq \infty$.

Exercise 4 (10 points). Show that

$$\left(\int_{\Omega_2} \left| \int_{\Omega_1} F(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \right|^p \, d\mathbf{y} \right)^{\frac{1}{p}} \leq \int_{\Omega_1} \left(\int_{\Omega_2} |F(\mathbf{x}, \mathbf{y})|^p \, d\mathbf{y} \right)^{\frac{1}{p}} \, d\mathbf{x}.$$

Exercise 5 (10 points). Deduce that if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, then $f \in L^r(\Omega)$ for every r between p and q . More precisely, write

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad \text{with } 0 \leq \alpha \leq 1$$

and prove that

$$\|f\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}^{\alpha} \|f\|_{L^q(\Omega)}^{1-\alpha}.$$