

## DIFFERENTIAL EQUATIONS (751873002, 114-2) - HOMEWORK 3

Return by April 10, 2026 (Friday) 23:59

Total marks: 50

**Special requirement.** All homework must be prepared by using L<sup>A</sup>T<sub>E</sub>X.

**Exercise 1** (10 points). Define  $f \in L^1_{\text{loc}}(\mathbb{R})$  as

$$f(x) = \begin{cases} x + a & \text{if } x > 0, \\ -x & \text{if } x < 0. \end{cases}$$

Determine whether the weak derivative of  $f$  exists or not for each  $a \in \mathbb{R}$ .

**Exercise 2** (10 points). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) := \text{sgn}(x)\sqrt{|x|}$  for all  $x \in \mathbb{R}$ , where

$$\text{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Show that the (order one) weak derivative  $f'$  exists, and compute it.

**Exercise 3** (10+10 points). Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) := H(x) + H(y)$  for all  $(x, y) \in \mathbb{R}^2$ , where  $H$  is the Heaviside function given by

$$H(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We denote the multiindices  $\alpha = (1, 1)$  and  $\beta = (1, 0)$ .

- (a) Prove that the weak derivatives  $\partial^\alpha f$  and  $\partial^{\alpha+\beta} f$  exist, and compute them.
- (b) Prove that the weak derivative  $\partial^\beta f$  does not exist. Compute its distributional derivative  $\partial^\beta f$ .

**Exercise 4** (10 points). Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) := H(x) - \text{sgn}(y)$  for all  $(x, y) \in \mathbb{R}^2$ . We denote the multiindex  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$ . Prove that the weak derivative  $\partial^\alpha f$  exists if and only if  $\alpha_1 \geq 1$  and  $\alpha_2 \geq 1$ .