

DIFFERENTIAL EQUATIONS (751873002, 113-2) - HOMEWORK 2

Return by March 13, 2025 (Thursday) 23:59

Total marks: 50

Special requirement. All homeworks must be prepared by using L^AT_EX.

Exercise 1 (10 points). Show that

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_{\Omega} f(\mathbf{x})g(\mathbf{x}) \, d\mathbf{x}$$

and

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_{\Omega} |f(\mathbf{x})g(\mathbf{x})| \, d\mathbf{x}.$$

Here f is not necessarily in $L^p(\Omega)$.

Exercise 2 (10 points). Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $h \in L^r(\Omega)$ for some $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Show that $fgh \in L^1(\Omega)$ and

$$\int_{\Omega} |f(\mathbf{x})g(\mathbf{x})h(\mathbf{x})| \, d\mathbf{x} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \|h\|_{L^r(\Omega)}.$$

Exercise 3 (10 points). ¹ Show that $\|\cdot\|_{L^p(\Omega)}^p$ defines a norm for each $0 < p < 1$. In addition, show that $\|\cdot\|_{L^p(\Omega)}$ does not define a norm for each $0 < p < 1$.

Exercise 4 (10 points). Let $n = 2$. Verify that Green's theorem is a special case of divergence theorem, see Exercise 1.0.20 for details.

Exercise 5 (10 points). Let A be a real symmetric matrix. Show that all its eigenvalue are positive if and only if

$$A\boldsymbol{\xi} \cdot \boldsymbol{\xi} \equiv \boldsymbol{\xi}^T A \boldsymbol{\xi} > 0 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

¹It is interesting to compare this exercise with Exercise 3 in Homework 1.