DIFFERENTIAL EQUATIONS (751873002, 113-2) - HOMEWORK 5

Return by April 10, 2025 (Thursday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10 points)**.** Let $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ be Banach spaces. Show that

$$\|\cdot\|_{W_1\cap W_2} := \|\cdot\|_{W_1} + \|\cdot\|_{W_2} \quad \text{and} \quad \|\cdot\|'_{W_1\cap W_2} := \max\{\|\cdot\|_{W_1}, \|\cdot\|_{W_2}\}$$

are norms, and they are equivalent. In addition, show that $(W_1 \cap W_2, \|\cdot\|_{W_1\cap W_2})$ is a Banach space, equivalently $(W_1 \cap W_2, \|\cdot\|'_{W_1\cap W_2})$ is a Banach spaces.

Exercise 2 (10 points)**.** Show the following Cauchy-Schwartz inequality:

$$|(u,v)| \le (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}} \quad \text{for all } u,v \in H.$$

In addition, show that the function $\|\cdot\|$ defined by $\|u\| := (u,u)^{\frac{1}{2}}$ for all $u \in H$ is a norm, which satisfies the parallelogram law:

$$\left\|\frac{u+v}{2}\right\|^2 + \left\|\frac{u-v}{2}\right\|^2 = \frac{1}{2}(\|u\|^2 + \|v\|^2) \quad \text{for all } u,v \in H. \tag{0.1}$$

Exercise 3 (10 points)**.** Let $(H, \|\cdot\|)$ be a normed space. Suppose that the norm $\|\cdot\|$ satisfies the parallelogram law (0.1). Define

$$(u,v) := \frac{1}{2}(\|u+v\|^2 - \|u\|^2 - \|v\|^2) \quad \text{for all } u,v \in H.$$

(1) Check that $(u,u) = \|u\|^2$, $(u,v) = (v,u)$, $(-u,v) = -(u,v)$ and $(u,2v) = 2(u,v)$ for all $u,v \in H$.
(2) Prove that $(u+v,w) = (u,w)+(v,w)$ for all $u,v,w \in H$. [Hint: use the parallelogram law successively with (i) $u = \tilde{u}$, $v = \tilde{v}$; (ii) $u = \tilde{u}+\tilde{w}$, $v = \tilde{v}+\tilde{w}$; and (iii) $u = \tilde{u}+\tilde{v}+\tilde{w}$, $v = \tilde{w}$]
(3) Prove that $(\lambda u, v) = \lambda(u,v)$ for all $\lambda \in \mathbb{R}$ and for all $u,v \in H$. [Hint: Consider first the case $\lambda \in \mathbb{N}$, then $\lambda \in \mathbb{Q}$, and finally $\lambda \in \mathbb{R}$]
(4) Conclude that $(\cdot,\cdot)$ is a scalar product on H.

Exercise 4 (10 points)**.** Show that $\|f\|_{L^p(\Omega)}$ satisfies the parallelogram law (0.1) if and only if $p = 2$. [Hint: Use functions with disjoint supports]

Exercise 5 (10 points)**.** Let Ω be any open set in $\mathbb{R}^n$, then we denote $H^m(\Omega) := W^{m,2}(\Omega)$ for each $m \in \mathbb{N}$. In this case, the norm reads

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} \|\partial^\alpha u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Use Exercise 3 to show that the corresponding scalar product is given by

$$(u,v)_{H^m(\Omega)} = \sum_{|\alpha|\le m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}.$$