DIFFERENTIAL EQUATIONS (751873002, 113-2) - HOMEWORK 6

Return by April 17, 2025 (Thursday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10 points)**.** For each vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ (identify as $n \times 1$ matrix), we define the juxtaposition $\boldsymbol{a} \otimes \boldsymbol{b} \in \mathbb{R}^{n \times n}$ (i.e. an $n \times n$ matrix with entries in $\mathbb{R}$) by $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\intercal$, where $\intercal$ denotes the transpose of the vector. Compute each entry $(\boldsymbol{a} \otimes \boldsymbol{b})_{ij}$ of the $n \times n$ matrix $\boldsymbol{a} \otimes \boldsymbol{b}$.

Exercise 2 (10 points)**.** Let $\boldsymbol{e}_j \in \mathbb{R}^n$ be the $j^{\rm th}$ column of the identity matrix Id_n. Show that

$$\sum_{k=1}^{n} \boldsymbol{e}_k \otimes \boldsymbol{e}_k = \mathrm{Id}_n.$$

Exercise 3 (10 points)**.** Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and consider the matrix $A := \mathrm{Id}_n + \boldsymbol{u} \otimes \boldsymbol{v}$, which is called the *rank-one perturbation of identity*. Determine the relation between $\boldsymbol{u}$ and $\boldsymbol{v}$ to guarantee A^{-1} exists, and compute A^{-1}.

Exercise 4 (10 points)**.** Let X and Y are normed spaces. One says that $\mathscr{L} : X \to Y$ is *bounded* (or *continuous*) if there is a constant $c \geq 0$ such that

$$\|\mathscr{L}u\|_Y \leq c\|u\|_X \quad \text{for all } u \in X.$$

Verify that

$$\|\mathscr{L}\|_{X \to Y} := \inf \{c \geq 0 : \|\mathscr{L}u\|_Y \leq c\|u\|_X \text{ for all } u \in X\} \equiv \sup_{u \neq 0} \frac{\|\mathscr{L}u\|_Y}{\|u\|_X}$$

is a norm.

Exercise 5 (10 points)**.** Let H be a complex vector space. We say that a mapping $(\cdot, \cdot) : H \times H \to \mathbb{C}$ is a complex inner product if

(1) **Linearity.** $(a_1u_1 + a_2u_2, v)_H = a_1(u_1, v)_H + a_2(u_2, v)_H$ for all $a_1, a_2 \in \mathbb{C}$ and for all $u_1, u_2, v \in H$;
(2) **Sesquilinearity.** $(u, a_1v_1 + a_2v_2)_H = \overline{a_1}(u, v_1)_H + \overline{a_2}(u, v_2)_H$ for all $a_1, a_2 \in \mathbb{C}$ and for all $u, v_1, v_2 \in H$;
(3) **Positive definiteness.** $(u, u) \geq 0$ for all $u \in H$ and $(u, u) = 0$ iff $u = 0$;
(4) **Skew symmetry.** $(u, v) = \overline{(v, u)}$ for all $u, v \in H$.

Show that $\|u\|_H := \sqrt{(u, u)_H}$ defines a norm. Determine the parallelogram law for $\|\cdot\|_H$.