PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 10

Return by: June 6, 2024 (Thursday) 16:00

Total marks: 50

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** Given any function $f : \mathbb{R} \to \mathbb{R}$. We say that f is odd (resp. even) if $f(-x) = -f(x)$ (resp. $f(-x) = f(x)$) for all $x \in \mathbb{R}$. Show that f can be *uniquely* decomposed as $f = f_{\rm even} + f_{\rm odd}$, where $f_{\rm even}$ is even and $f_{\rm odd}$ is odd.

Exercise 2 (10 points)**.** Compute the Fourier sine series and Fourier cosine series of the constant function $f(x) = 1$ on the interval $(0, \pi)$.

It is easy to see that $\{e^{\mathbf{i}\boldsymbol{j}\cdot\boldsymbol{x}}\}_{\boldsymbol{j}\in\mathbb{Z}^n}$ forms an orthonormal set in $L^2(Q)$ with respect to the normalized inner product

$$(f, g)_{L^2(Q)} := \frac{1}{|Q|} \int_Q f(\boldsymbol{x}) \overline{g(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x}$$

where $Q := [-\pi, \pi]^n$. In class, we proved that $\{e^{\mathbf{i}\boldsymbol{j}\cdot\boldsymbol{x}}\}_{\boldsymbol{j}\in\mathbb{Z}^n}$ is indeed a Hilbert basis of $L^2(Q)$.

Exercise 3 (10 points)**.** Let $T > 0$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a function with period $2T$ on each variable. Compute the Fourier coefficient $\hat{f}(\boldsymbol{k})$ of the Fourier series

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \hat{f}(\boldsymbol{k}) e^{\mathbf{i}\frac{\pi}{T}\boldsymbol{k}\cdot\boldsymbol{x}} \quad \text{on the square } (-T, T)^n.$$

Exercise 4 (10 points)**.** For the case when $n = 1$, express the Fourier series in Exercise 3 of the form

$$f(x) = \frac{1}{2}B_0 + \sum_{k=1}^{\infty} \left(A_k \sin\left(\frac{\pi}{T}kx\right) + B_k \cos\left(\frac{\pi}{T}kx\right) \right) \quad \text{on the square } (-T, T)^n.$$

Exercise 5 (10 points)**.** Find a necessary and sufficient condition in terms of Fourier coefficient $\hat{f}(\boldsymbol{k})$ in Exercise 3 so that f is real-valued.