PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 1

Return by: March 7, 2024 (Thursday) 16:00

Total marks: 50

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** For each $1 < p < \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1$, show the following inequality:

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \quad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

Use this to conclude the Hölder's inequality

$$\int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

[Hint: One way to show this is using the concavity of logarithmic function on $(0, \infty)$]

Exercise 2 (10 points)**.** Show that $\| \cdot \|_{L^p(\Omega)} : L^p(\Omega) \to \mathbb{R}$ defines a norm. [Hint: Use the convexity of the mapping $t \mapsto t^p$ for each $1 \leq p < \infty$]

Exercise 3 (10 points)**.** Given any $f \in L^p(\Omega)$, show that

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x},$$

as well as

$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^{p'}(\Omega)}=1} \int_\Omega |f(\boldsymbol{x})g(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}.$$

Exercise 4 (10 points)**.** In fact, the result in Exercise 3 holds true for any (measurable) f (not necessarily in $L^p(\Omega)$). Use this fact to show

$$\left(\int_{\Omega_2} \left| \int_{\Omega_1} F(\boldsymbol{x}, \boldsymbol{y}) \,\mathrm{d}\boldsymbol{x} \right|^p \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}} \leq \int_{\Omega_1} \left(\int_{\Omega_2} |F(\boldsymbol{x}, \boldsymbol{y})^p| \,\mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}} \mathrm{d}\boldsymbol{x}.$$

Exercise 5 (10 points)**.** Let Ω be an open set in $\mathbb{R}^n$. Show that $C_c^\infty(\Omega)$ is not dense in $L^\infty(\Omega)$.