PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 2

Return by: March 14, 2024 (Thursday) 16:00

Total marks: 50 (10 bonus)

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** Deduce that if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \le p \le \infty$ and $1 \le q \le \infty$, then $f \in L^r(\Omega)$ for every r between p and q. More precisely, write

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad \text{with } 0 \le \alpha \le 1$$

and prove that

$$\|f\|_{L^r(\Omega)} \le \|f\|^{\alpha}_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)}.$$

Exercise 2 (10 points)**.** Find the general solution $u(t,x)$ of $\partial_t u + \partial_x u = 1$.

Exercise 3 (10 points)**.** Find the general solution $u(t,x)$ of $\partial_t u + c\partial_x u + ku = 0$, where c and k are constants.

Exercise 4 (10 points)**.** Find the general solution $u(t,x)$ of $\partial_t u + 2\partial_x u + (2t - x)u = 2t^2 + 3tx - 2x^2$.

Exercise 5 (10 points)**.** Verify the function γ given in

$$\gamma(t) = \begin{cases} 0 & \text{for } t < 3, \\ \left(\frac{2(t^2-9)}{5}\right)^{5/4} & \text{for } t \ge 3, \end{cases}$$

is in $C^1(\mathbb{R})$ and it solves $\gamma'(t) = t(\gamma(t))^{1/5}$ for all $t \in \mathbb{R}$ and $\gamma(3) = 0$.

Exercise 6 (bonus: 10 points)**.** Using computer software to solve Exercise 2.2.5.