PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 4

Return by: March 28, 2024 (Thursday) 16:00

Total marks: 50

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** The solution of the 1-dimensional wave equation can be used to approximate the pointwise displacement of the vibrating of an (infinitely) long string. The sound speed c is given by $c = \sqrt{T/\rho}$, where T is the tension of string and ρ is the density of the string. Consider the string with initial position

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases}$$

and initial velocity $\psi(x) \equiv 0$. This modeling the "three-finger" pluck, with all three fingers removed at once. Note that $\phi \in C^0(\mathbb{R})$ but not differentiable at $x = 0, \pm a$. Compute the weak solution from the d'Alembert formula. [Hint: Consider the cases $t < \frac{a}{c}$, $t = \frac{a}{c}$, $t > \frac{a}{c}$]

Exercise 2 (10 points)**.** Let $\phi(x) \equiv 0$ and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a. \end{cases}$$

In this case, ψ is not differentiable at $x = \pm a$. Compute the solution u from the d'Alembert formula.

Exercise 3 (10 points)**.** Find a solution of the initial-boundary value problem

$$\begin{cases} \partial_t^2 u(t,x) - c^2 \partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty) \times (0,\infty), \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x \in (0,\infty), \\ u(t,0) = 0 & \text{for all } t \in (0,\infty) \end{cases}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ with compatibility condition $\phi(0) = \phi'(0) = \phi''(0) = \psi(0) = \psi'(0) = 0$. Express u in terms of ϕ and ψ.

Exercise 4 (10 points)**.** Find a solution of the initial-boundary value problem

$$\begin{cases} \partial_t^2 u(t,x) - c^2 \partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty) \times (0,\infty), \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x) & \text{for all } x \in (0,\infty), \\ \partial_x u(t,0) = 0 & \text{for all } t \in (0,\infty) \end{cases}$$

for $\phi \in C^2((0,\infty))$ and $\psi \in C^1((0,\infty))$ with compatibility condition $\phi(0) = \phi'(0) = \phi''(0) = \psi(0) = \psi'(0) = 0$. Express u in terms of ϕ and ψ.

Exercise 5 (10 points)**.** Find a weak solution of the initial-boundary value problem

$$\begin{cases} \partial_t^2 u(t,x) - c^2 \partial_x^2 u(t,x) = 0 & \text{for all } (t,x) \in (0,\infty) \times (0,\infty), \\ u(0,x) = 0, \quad \partial_t u(0,x) = V & \text{for all } x \in (0,\infty), \\ \partial_t u(t,0) + a \partial_x u(t,0) = 0 & \text{for all } t \in (0,\infty), \end{cases}$$

where V, a, c are positive constants with $a > c$.