PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 5

Return by: April 11, 2024 (Thursday) 16:00

Total marks: 50 (10 bonus)

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** Suppose that $g \in C^2$ is radially symmetric, i.e. $g(\boldsymbol{y}) = g(r)$ with $r = |\boldsymbol{y}|$. Show that

$$\Delta_{\boldsymbol{y}} g = r^{1-n}\partial_r(r^{n-1}\partial_r g) = \partial_r^2 g + \frac{n-1}{r}\partial_r g.$$

Exercise 2 (10 points)**.** Let $f \in C^2((0,\infty)\times\mathbb{R}^3)$, $\psi \in C^2(\mathbb{R}^3)$ and $\phi \in C^3(\mathbb{R}^3)$. Find the unique C^2-solution of the following initial-value problem:

$$\begin{cases} \partial_t^2 u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = f(t,\boldsymbol{x}) & \text{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^3, \\ u(0,\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathbb{R}^3, \end{cases}$$

using Duhamel principle.

Exercise 3 (10 points)**.** Let $f \in C^2((0,\infty)\times\mathbb{R}^2)$, $\psi \in C^2(\mathbb{R}^2)$ and $\phi \in C^3(\mathbb{R}^2)$. Find the unique C^2-solution of the following initial-value problem:

$$\begin{cases} \partial_t^2 u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = f(t,\boldsymbol{x}) & \text{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^2, \\ u(0,\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathbb{R}^2, \end{cases}$$

using Duhamel principle.

Exercise 4 (20 points)**.** Consider the initial-value problem

$$\begin{cases} \partial_t^2 u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = 0 & \text{for } (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^5, \\ u(0,\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathbb{R}^5. \end{cases}$$

Let

$$M_u(t,\boldsymbol{x},r) = \frac{1}{\omega_n}\int_{\partial B_1(0)} h(\boldsymbol{x}+r\hat{\boldsymbol{z}})\,\mathrm{d}\hat{\boldsymbol{z}}.$$

Set $N_u(t,\boldsymbol{x},r) = r^2\partial_r M_u(t,\boldsymbol{x},r) + 3rM_u(t,\boldsymbol{x},r)$.

(a) Show that $N_u(t,\boldsymbol{x},r)$ is a solution of $\partial_t^2 N_u = c^2\partial_r^2 N_u$ and find N_u from its initial data in terms of $M_f(\boldsymbol{x},r)$ and $M_g(\boldsymbol{x},r)$.

(b) Show that

$$u(t,\boldsymbol{x}) = \lim_{r\to 0}\frac{N_u(t,\boldsymbol{x},r)}{3r} = \left(\frac{1}{3}t^2\partial_t + t\right)M_g(ct,\boldsymbol{x}) + \partial_t\left(\frac{1}{3}t^2\partial_t + t\right)M_f(ct,\boldsymbol{x}).$$

Exercise 5 (10 points)**.** Solve

$$\begin{cases} \partial_t^2 u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = 0 & \text{for } (t,\boldsymbol{x}) \in (0,\infty) \times \mathbb{R}^4, \\ u(0,\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathbb{R}^4, \end{cases}$$

using the Hadamard's method of descent.