PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 6

Return by: April 11, 2024 (Thursday) 16:00

Total marks: 50

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** Let X be a space with a sequence of norms $\{\|\cdot\|_N\}_{N\in\mathbb{N}}$. We define the mapping $\mathsf{d} : X \times X \to \mathbb{R}$ by

$$\mathsf{d}(x,y) := \sum_{N\in\mathbb{N}} 2^{-N} \frac{\|x-y\|_N}{1+\|x-y\|_N} \quad \text{for all } x,y \in X.$$

Show that the mapping $\mathsf{d} : X \times X \to \mathbb{R}$ above is a metric on X, and thus (X,d) forms a metric space.

Exercise 2 (10 points)**.** Take $n = 1$ and $\Omega = \mathbb{R}$. Let $\phi \in \mathscr{D}(\mathbb{R}) \equiv C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\phi) \subset [0,1]$ and $\phi > 0$ in $(0,1)$. Define

$$\psi_m(x) := \phi(x-1) + \frac{1}{2}\phi(x-2) + \cdots + \frac{1}{m}\phi(x-m).$$

Show that $\{\psi_m\}$ is a Cauchy sequence in $(\mathscr{D}(\mathbb{R}), \mathsf{d}_{\mathscr{D}(\mathbb{R})})$, where $\mathsf{d}_{\mathscr{D}(\mathbb{R})}$ is the metric given in the lecture note. Using this to conclude that $(\mathscr{D}(\mathbb{R}), \mathsf{d}_{\mathscr{D}(\mathbb{R})})$ is not complete.

Exercise 3 (10 points)**.** Prove that for every $c \in \mathbb{R}$ one has

$$(e^{-c|x|})' = -ce^{-cx}H(x) + ce^{cx}H(-x) \quad \text{in distribution sense.}$$

Exercise 4 (10 points)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \ln|x| - x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that f is a continuous function and compute its distributional derivative f'.

Exercise 5 (10 points)**.** Let $n = 1$, and let δ_a be the Dirac measure at $a \in \mathbb{R}$. Show that $T = \sum_{j=1}^\infty \partial^j \delta_j$ is a distribution.