PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 7

Return by: May 16, 2024 (Thursday) 16:00

Total marks: 50

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** Show the following Cauchy-Schwartz inequality:

$$|(u,v)| \leq (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}} \quad \text{for all } u,v \in H.$$

In addition, show that the function $\|\cdot\|$ defined by $\|u\| := (u,u)^{\frac{1}{2}}$ for all $u \in H$ is a norm, which satisfies the parallelogram law

$$\left\|\frac{u+v}{2}\right\|^2 + \left\|\frac{u-v}{2}\right\|^2 = \frac{1}{2}(\|u\|^2 + \|v\|^2) \quad \text{for all } u,v \in H. \tag{0.1}$$

Exercise 2 (10 points)**.** Let $(H, \|\cdot\|)$ be a normed space. Suppose that the norm $\|\cdot\|$ satisfies the parallelogram law (0.1). Define

$$(u,v) := \frac{1}{2}(\|u+v\|^2 + \|u\|^2 - \|v\|^2) \quad \text{for all } u,v \in H.$$

Prove that $(\cdot,\cdot)$ is a scalar product on H. The hints are given in lecture note.

Exercise 3 (10 points)**.** Let $1 \leq p \leq \infty$. Show that $L^p(\mathbb{R}^n)$ is a Hilbert space if and only if $p = 2$.

Exercise 4 (5 points)**.** For each vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ (identify as $n \times 1$ matrix), we define the juxtaposition $\boldsymbol{a} \otimes \boldsymbol{b} \in \mathbb{R}^{n\times n}$ (i.e. an $n \times n$ matrix with entries in $\mathbb{R}$) by $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\intercal$, where $\intercal$ denotes the transpose of the vector. Compute each entry $(\boldsymbol{a} \otimes \boldsymbol{b})_{ij}$ of the $n \times n$ matrix $\boldsymbol{a} \otimes \boldsymbol{b}$.

Exercise 5 (5 points)**.** Let $\boldsymbol{e}_j \in \mathbb{R}^n$ be the j^{th} column of the identity matrix Id_n. Show that

$$\sum_{k=1}^{n} \boldsymbol{e}_k \otimes \boldsymbol{e}_k = \mathrm{Id}_n.$$

Exercise 6 (10 points)**.** Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and consider the matrix $A := \mathrm{Id}_n + \boldsymbol{u} \otimes \boldsymbol{v}$, which is called the *rank-one perturbation of identity*. Determine the relation between $\boldsymbol{u}$ and $\boldsymbol{v}$ to guarantee A^{-1} exists, and compute A^{-1}.