PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 8

Return by: May 23, 2024 (Thursday) 16:00

Total marks: 50

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** Let X and Y are Banach spaces. Verify that the function $\|\cdot\|_{X\to Y}$ defined by

$$\|L\|_{X\to Y} := \sup_{X\ni u\not\equiv 0} \frac{\|Lu\|_Y}{\|u\|_X} \quad \text{for all bounded linear operator } L : X \to Y$$

defined a norm.

Exercise 2 (10 points)**.** Let X and Y are Banach spaces. The previous exercise showed that $(\mathscr{B}(X,Y).\|\cdot\|_{X\to Y})$ defines a normed space. Show that $(\mathscr{B}(X,Y).\|\cdot\|_{X\to Y})$ is also a Banach space.

Exercise 3 (10 points)**.** Let I be a bounded interval in $\mathbb{R}$. Show that there exists a constant C, depending on the length of the interval $|I| < \infty$, such that

$$\|u\|_{L^2(I)} \le C\|u'\|_{L^2(I)} \quad \text{for all } u \in H_0^1(I).$$

Exercise 4 (10 points)**.** Let Ω be a bounded open set. Proof the Poincaré inequality:

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega).$$

Exercise 5 (10 points)**.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $f \in H^{-1}(\Omega)$. Suppose that $k^2 < \lambda_1$ and we define the functional

$$F(v) = \frac{1}{2}\left(\|\nabla v\|_{L^2(\Omega)}^2 - k^2\|v\|_{L^2(\Omega)}^2\right) + \langle f, v\rangle_{H^{-1}(\Omega)\otimes H_0^1(\Omega)} \quad \text{for all } v \in H_0^1(\Omega).$$

Show that there exists a unique $u \in H_0^1(\Omega)$ such that $F(u) \le F(v)$ for all $v \in H_0^1(\Omega)$. [Hint: Read the lecture note carefully]