PARTIAL DIFFERENTIAL EQUATIONS (701925001, 751944001, 112-2) - HOMEWORK 9

Return by: June 6, 2024 (Thursday) 16:00

Total marks: 50 (with 10 extra points)

Note. One should try to solve all problems in the lecture note. Here I only choose some of them in this homework.

Exercise 1 (10 points)**.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Given $f \in H^{-1}(\Omega)$, by using Theorem 3.4.8 in the lecture note with $k = 0$, one can find a unique solution $u \in H_0^1(\Omega)$ of the problem

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \langle f, v \rangle_{H^{-1}(\Omega) \oplus H_0^1(\Omega)} \quad \text{for all } v \in H_0^1(\Omega).$$

Then one can define the linear operator $(-\Delta)_{\rm Dir}^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ by $(-\Delta)_{\rm Dir}^{-1} f := u$. By doing some suitable identifications, show that $(-\Delta)_{\rm Dir}^{-1} \in \mathcal{K}(L^2(\Omega))$.

Exercise 2 (10 points)**.** By choosing $T = (-\Delta)_{\rm Dir}^{-1}$ and $H = L^2(\Omega)$, where $(-\Delta)_{\rm Dir}^{-1} \in \mathcal{K}(L^2(\Omega))$ is the operator in Exercise 1, show that there exists $u_* \in H_0^1(\Omega)$ such that

$$\inf_{0 \not\equiv u \in H_0^1(\Omega)} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} = \frac{\|\nabla u_*\|_{L^2(\Omega)}^2}{\|u_*\|_{L^2(\Omega)}^2}.$$

In addition, show that there exists a constant $c \neq 0$ such that $u_* = c\phi_1$.

Exercise 3 (10 points)**.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\{\lambda_j\}_{j=1}^\infty$ be eigenvalues of the Dirichlet Laplacian $-\Delta$ with corresponding eigenfunctions $\{\phi_j\}_{j=1}^\infty$ as in Theorem 3.6.4 of the lecture note. Show that

$$\lambda_j = \min \left\{ \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} : 0 \not\equiv u \in H_0^1(\Omega) \text{ with } u \perp \phi_i \text{ for all } i = 1, \cdots, j-1 \right\} \quad \text{for } j = 2, 3, 4, \cdots$$

Here $u \perp \phi_i$ means $(u, \phi_i)_{L^2(\Omega)} = 0$.

Exercise 4 (10 points)**.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\{\lambda_j\}_{j=1}^\infty$ be eigenvalues of the Dirichlet Laplacian $-\Delta$ with corresponding eigenfunctions $\{\phi_j\}_{j=1}^\infty$ as in Theorem 3.6.4 of the lecture note. Show that

$$\lambda_j = \min_{F_j \subset H_0^1(\Omega), \dim F_j = j} \left(\max_{u \in F_j} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} \right),$$

where $\min_{F_j \subset H_0^1(\Omega), \dim F_j = j}$ means that the minimum is taken over all finite dimensional vector space $F_j \subset H_0^1(\Omega)$ with $\dim F_j = j$.

Exercise 5 (10 points)**.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $u \in H_0^1(\Omega)$. If $k^2 \neq \lambda_j$ for all $j \in \mathbb{N}$, show that

$$u \equiv 0 \text{ in } \Omega \quad \text{if and only if} \quad (\Delta + k^2) u = 0 \text{ in } \Omega \text{ (give a suitable formulation).}$$

Exercise 6 (10 points)**.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, let $f \in H^{-1}(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $k \geq 0$ with k^2 is not an eigenvalue of the Dirichlet Laplacian $-\Delta$ as in Theorem 3.6.4 of the lecture note. Show that there exists a unique $u \in H^1(\Omega)$ satisfies

$$(\Delta + k^2)u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = g.$$