FROM LINEAR ALGEBRA TO FUNCTIONAL ANALYSIS (701853001, 115-1) - HOMEWORK 1

Return by October 1, 2026 (Thursday) 23:59

Total marks: 50

Special requirement. All homework must be prepared by using LaTeX.

Exercise 1 (10 points)**.** Given a basis of $\{A^{(1)}, \cdots, A^{(mn)}\} \subset \mathbb{C}^{m\times n}$, construct an orthonormal basis of $\mathbb{C}^{m\times n}$ accordingly. (**Hint.** Let $A, B \in \mathbb{C}^{m\times n}$, show that $A - (A, \hat{B})\hat{B}$ is orthogonal to B with $\hat{B} := B/\|B\|_{\mathbb{C}^{m\times n}}$)

Exercise 2 (10 points)**.** Let $A, B \in \mathbb{C}^{n\times n}$, show that

(a) $\|A + B\| \leq \|A\| + \|B\|$,

(b) $\|AB\| \leq \|A\|\|B\|$.

Exercise 3 (10 points)**.** All norms on finite-dimensional vector spaces V are equivalent, that is, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on V, then there exists a constant $c > 0$ such that

$$c\|u\|_1 \leq \|u\|_2 \leq c^{-1}\|u\|_1 \quad \text{for all } u \in V.$$

Exercise 4 (10 points)**.** Construct a sequence of functions $\{f_k : [0, 1] \to \mathbb{C}\}_{k\in\mathbb{N}}$ such that f_k converges strongly in $L^1(0, 1)$, but not converges strongly in $L^\infty(0, 1)$.

Exercise 5 (10 points)**.** Show the following Cauchy-Schwartz inequality:

$$|(u, v)| \leq (u, u)^{\frac{1}{2}}(v, v)^{\frac{1}{2}} \quad \text{for all } u, v \in H.$$

In addition, show that the function $\|\cdot\|$ defined by $\|u\| := (u, u)^{\frac{1}{2}}$ for all $u \in H$ is a norm, which satisfies the parallelogram law:

$$\left\|\frac{u + v}{2}\right\|^2 + \left\|\frac{u - v}{2}\right\|^2 = \frac{1}{2}(\|u\|^2 + \|v\|^2) \quad \text{for all } u, v \in H.$$