STATISTICS "FOR DEPARTMENT OF MATHEMATICAL SCIENCES" (701007001, 114-2) - HOMEWORK 2

Return by April 15, 2026 (Wednesday) 15:00

Total marks: 50

Note. Using LaTeXto prepare your homework is encouraged but not required. If you do so, please print it out and submit a hard copy.

Exercise 1 (10 points)**.** Suppose that $X_1, X_2, \cdots, X_N$ are random sample with $\mathcal{N}(\mu, \sigma^2)$. Using Theorem 3.1.14, we know that

$$\frac{N-1}{\sigma^2} S_u^2 \sim \chi^2_{N-1} \quad \text{where} \quad S_u^2 := \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \overline{X})^2 \text{ and } \overline{X} = \frac{1}{N} \sum_{i=1}^{N} X_i,$$

therefore $\mathrm{var}(\frac{N-1}{\sigma^2} S_u^2) = 2(N-1)$. We now consider an estimate of σ^2 of the form

$$S^2 = c \sum_{i=1}^{N} (X_i - \overline{X})^2 \quad \text{for some } c > 0.$$

Compute the mean squared error (MSE) of S^2.

Exercise 2. Let $\alpha > 0$ and let $X_1, X_2, \cdots, X_N$ are random sample with

$$\mathbb{P}(X_i \le y) = \begin{cases} 0 & , y \le 0, \\ y^\alpha / \theta^\alpha & , 0 \le y \le \theta, \\ 1 & y \ge \theta. \end{cases}$$

(a) (10 points)**.** Using the fact $\mathbb{E}Y = \int_0^\infty \mathbb{P}(Y > y)\,\mathrm{d}y$ and $\mathbb{E}Y^2 = 2\int_0^\infty y\mathbb{P}(Y > y)\,\mathrm{d}y$ for all nonnegative random variable Y, compute the expectation and variance of

$$\hat{\theta} = c \max_{1 \le i \le N} X_i$$

for any $c > 0$.

(b) (5 points)**.** Compute the mean squared error (MSE) of $\hat{\theta}$.

(c) (5 points)**.** Determine the value of $c > 0$ so that $\hat{\theta}$ is unbiased.

(c) (5 points)**.** Determine the value of $c > 0$ so that $\hat{\theta}$ minimize the MSE.

(d) (5 points)**.** Compute the MSE of the estimator $\hat{\theta}' = d\overline{X}$ of θ for all $d > 0$.

(e) (5 points)**.** Determine the value of $d > 0$ so that the estimator $\hat{\theta}' = d\overline{X}$ is unbiased.

(f) (5 points)**.** Determine the value of $d > 0$ so that $\hat{\theta}'$ minimize the MSE.